11.2 Abelian Groups of prime power order

Before we cover the classification of abelian groups, we must first understand direct products and abelian groups whose order is a prime power.

Proposition 11.2.1. *Let C_n and C_m be finite cyclic groups such that* $\gcd(m,n) = 1$*. Then* $C_n \times C_m \cong C_{nm}$*.*

Proof. Let $m, n \in \mathbb{N}$ be such that $\gcd(m,n) = 1$. The group C_{nm} is generated by a single element ρ whose order is nm. Now C_{nm} is cyclic and so it is abelian. Hence every subgroup of C_{nm} is normal.
Let $G = \langle \rho^m \rangle \leq C_{nm}$ and $H = \langle \rho^n \rangle \leq C_{nm}$. Since $o(\rho^m) = n$ and $o(\rho^n) = m$ we have $G \cong C_n$ and $H \cong C_m$.
Every subgroup of C_{nm} is normal. Therefore $G \trianglelefteq C_{nm}$ and $H \trianglelefteq C_{nm}$. Moreover, if $\rho^k \in G \cap H$, then k must be a multiple of m and a multiple of n, which implies that $\rho^k = e$. Hence $G \cap H = \langle e \rangle$.
Hence GH is an internal direct product, so by Theorem 11.1.3,

$$GH \cong G \times H \cong C_n \times C_m.$$

Finally, we note that since $GH \cong C_n \times C_m$, we have $|GH| = nm$. Moreover, $GH \leq C_{nm}$ and $|C_{nm}| = nm$. Therefore $GH = C_{nm}$. Hence $C_{nm} \cong C_n \times C_m$. □

Proposition 11.2.2. *Let C_n and C_m be finite cyclic groups such that* $\gcd(m,n) \neq 1$*. Then* $C_n \times C_m \ncong C_{nm}$*.*

Proof. Suppose, for a contradiction, that $\theta : C_{nm} \to C_n \times C_m$ is an isomorphism. Now C_{nm} is cyclic, so $C_{nm} = \langle g \rangle$ for some element g of order nm. Let $\theta(g) = (a,b)$, for some $a \in C_n$ and $b \in C_m$. Since θ is an isomorphism, it must be the case that $o((a,b)) = o(g) = nm$.
Let $d = \gcd(m,n) > 1$, and write $m = rd$ and $n = sd$. Since $d > 1$ we have that $ms < mn$. Now $(a,b)^{ms} = (a^{rds}, b^{rds}) = ((a^{sd})^r, (b^{rd})^s) = ((a^n)^r, (b^m)^s) = (e,e)$. Thus, $o((a,b)) \leq ms < nm$. A contradiction. □

Example 11.2.3. Consider the following examples.

- $C_{80} = C_{2^4 \cdot 5} \cong C_{2^4} \times C_5$ (By Proposition 0.1.1)
- $C_7 \times C_{5^2} \cong C_{7 \cdot 5^2} = C_{175}$ (By Proposition 0.1.1)
- $C_2 \times C_2 \ncong C_4$ (By Proposition 0.1.2)
- $C_2 \times C_{2^2} \times C_{2^3}$ and $C_{2^3} \times C_{2^3}$ and C_{2^6} are all nonisomorphic. (By Proposition 0.1.2)

We know the structure of all finite abelian p-groups.

Theorem 11.2.4. *Let p be a prime and let G be a finite abelian group of order p^n. Then there are natural numbers $e_1, e_2, \ldots, e_t$ such that G is isomorphic to a direct product*

$$C_{p^{e_1}} \times C_{p^{e_2}} \times \cdots \times C_{p^{e_t}},$$

where $p^{e_1} p^{e_2} \cdots p^{e_t} = p^n$, and this factorisation is unique up to reordering of the factors.

Proof not covered in lectures. The proof is beyond the scope of this course, but for those interested it can be found in Section 11.3.

Example 11.2.5. Here we list (up to isomorphism) all abelian groups of order $8 = 2^3$.

Shape	Decomposition of 2^3	Group	Remarks
(3)	2^3	C_{2^3}	$= C_8$
$(2, 1)$	$2^2 \cdot 2^1$	$C_{2^2} \times C_2$	$\cong C_2 \times C_{2^2}$
$(1, 1, 1)$	$2^1 \cdot 2^1 \cdot 2^1$	$C_2 \times C_2 \times C_2$	

Hence, up to isomorphism there are 3 abelian groups of order 8.