1 Permutations

1.1 An introduction to permutations

Recall that a *permutation* of a set X is a bijection from X to X. The collection of all permutations of X is written $\mathrm{Sym}(X)$. If X is just the set $\{1, 2, \dots, n\}$ then we write S_n instead of $\mathrm{Sym}(X)$.

Example 1.1.1. How large is S_n? Well, we have n choices for the image of 1. For the image of 2 we have only $n-1$ choices (since bijections are one-to-one). For the image of 3 we have only $n-2$ choices, etc. Finally, for the image of n we have only one choice remaining. Therefore:

$$|S_n| = n!$$

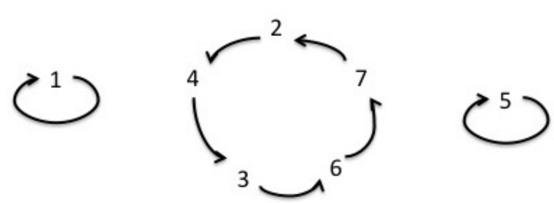
In fact the same argument tells us that there are $|X|!$ elements in $\mathrm{Sym}(X)$.

There are many different notations for permutations, but we will start to introduce a very powerful notation called *cycle notation*.

Example 1.1.2. Consider the following permutation σ of the set $X = \{1, 2, 3, 4, 5, 6, 7\}$:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 6 & 3 & 5 & 7 & 2 \end{array}$$

First of all notice that the top row lists the elements of X in some order. Because σ is a bijection, the second row can have no repeats (one-to-one) and must include all of X (onto) so the second row is just X again but in a different order. This is why people often speak of permutations as being a reordering of a set.
This notation is clear, but it is too cumbersome to use in a powerful theory like group theory.
Let's draw a picture of this permutation instead:



We can see that σ can be *broken down into cycles*. This is the key observation for our new notation. We can write the permutation σ as:

$$\sigma = (1)(2\,4\,3\,6\,7)(5)$$

Each bracket represents a cycle, so 1 is in cycle on its own, there's a cycle $2 \to 4 \to 3 \to 6 \to 7 \to 2$ and 5 is in a cycle on its own.
We don't bother writing down the cycles of length one (although we did in the example above to get used to the notation). So, we will usually write $\sigma = (2\,4\,3\,6\,7)$. Since we know that $\sigma \in S_7$ we can tell (because they are missing) that 1 and 5 are fixed by σ.

Example 1.1.3. Here are some more permutations in cycle notation.

(i) The permutation $\tau = (1\,2\,3\,4\,5\,6\,7\,8\,9)$ of the set $\{1, \dots, 9\}$ sends:

$$1 \to 2 \to 3 \to 4 \to 5 \to 6 \to 7 \to 8 \to 9 \to 1$$

It is the permutation:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \end{array}$$

See how much neater the cycle notation is? Wait until you start multiplying permutations—then you will really love it.

(ii) The permutation $\rho = (5\,2\,3)(7\,8\,1)(9\,4)$ of the set $\{1, \dots, 9\}$ sends:

$$5 \to 2 \to 3 \to 5 \quad \text{and} \quad 7 \to 8 \to 1 \to 7 \quad \text{and} \quad 9 \to 4 \to 9 \quad \text{and} \quad 6 \to 6$$

It is the permutation:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 3 & 5 & 9 & 2 & 6 & 8 & 1 & 4 \end{matrix}$$

(How do we know that $6 \to 6$? Because $\rho \in S_9$ and 6 is missing from all the cycles.)

How to apply permutations to elements of X. Recall that a permutation σ of X is a bijective function from X to X. How do we find the image $\sigma(x)$ of some $x \in X$ using the cycle notation?
Well first of all, we don't write $\sigma(x)$ because we are using "(" and ")" to denote cycles—instead we just write σx.

Definition 1.1.4. Let $a_1, a_2, \dots, a_r$ be **distinct** elements of a set X. Then

$$\sigma = (a_1\, a_2\, \dots a_r)$$

is the permutation sending

$$a_1 \to a_2 \to \cdots \to a_r \to a_1$$

and fixing everything else in $X \setminus \{a_1, \dots, a_r\}$. Such a permutation is called an *r-cycle* or a *cycle of length r*.

Definition 1.1.5. A collection of cycles in S_n is called a collection of *disjoint cycles* if no element of $\{1, 2, \dots, n\}$ appears in more than one of the cycles.

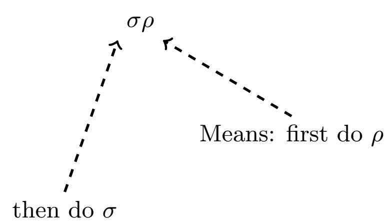
Example 1.1.6. Cycles $(1\,2\,3), (4\,7\,5)$ are disjoint, but cycles $(1\,2\,3), (4\,3\,7\,5)$ are not.

Remark 1.1.7. Notice that the cycles $(1\,2\,3)$ and $(3\,1\,2)$ and $(2\,3\,1)$ are all equal. There is more than one way to write down the same cycle.

1.2 Products of permutations

Definition 1.2.1. Let σ and ρ be permutations of the same set X, and recall that σ and ρ are bijective functions from X to X.
The *product* of the permutations σ and ρ is the function composition $\sigma \circ \rho$ that you have seen in your first year. Usually we don't bother writing the symbol $\circ$.

Notice that:



Warning! Some books do this the opposite way round (although most agree with the way we do it here).

Example 1.2.2. Here are some examples of products of permutations. You can calculate their product the long way (writing out $1, 2, \ldots$ then drawing the arrows, finding the images, then converting back to cycle notation) or you can do it quickly by keeping within the cycle notation

(i) If $\sigma = (1\,2\,3)$ and $\rho = (1\,5\,3\,4)$ are permutations in S_6, then:

(pick any element in set to start e.g. 1)
$\sigma\rho 1 = \sigma$ applied to $(\rho 1) = \sigma 5 = 5$ so $\sigma\rho$ looks like $(1\,5\,\ldots)\cdots$
$\sigma\rho 5 = \sigma$ applied to $(\rho 5) = \sigma 3 = 1$ so $\sigma\rho$ looks like $(1\,5)\cdots$
(we've finished a cycle so we just pick the next unknown)
$\sigma\rho 2 = \sigma 2 = 3$ so $\sigma\rho$ looks like $(1\,5)(2\,3\,\ldots)\cdots$
$\sigma\rho 3 = \sigma 4 = 4$ so $\sigma\rho$ looks like $(1\,5)(2\,3\,4\,\ldots)\cdots$
$\sigma\rho 4 = \sigma 1 = 2$ so $\sigma\rho$ looks like $(1\,5)(2\,3\,4)\cdots$
(we've finished a cycle so we just pick the next unknown)
$\sigma\rho 6 = \sigma 6 = 6$ so $\sigma\rho$ looks like $(1\,5)(2\,3\,4)(6)\cdots$
(there are no unknowns left so we are done)

Hence $\sigma\rho = (1\,5)(2\,3\,4)$ is a permutation in S_6.

(ii) If $\sigma = (1\,2)$ and $\rho = (2\,3)$ and $\tau = (3\,5)$ are permutations in S_5 then:

$\sigma\rho\tau 1 = \sigma\rho 1 = \sigma 1 = 2$ so $\sigma\rho\tau$ looks like $(1\,2\,\ldots)\cdots$
$\sigma\rho\tau 2 = \sigma\rho 2 = \sigma 3 = 3$ so $\sigma\rho\tau$ looks like $(1\,2\,3\,\ldots)\cdots$
$\sigma\rho\tau 3 = \sigma\rho 5 = \sigma 5 = 5$ so $\sigma\rho\tau$ looks like $(1\,2\,3\,5\,\ldots)\cdots$
$\sigma\rho\tau 5 = \sigma\rho 3 = \sigma 2 = 1$ so $\sigma\rho\tau$ looks like $(1\,2\,3\,5)\cdots$
(there are no unknowns left so we are done)

Hence $\sigma\rho\tau = (1\,2\,3\,5)$ is a permutation in S_5.

Question 1.2.3. A question to get you thinking about what you've seen so far.

(i) Let $\tau = (3\,5)$ and $\sigma = (2\,4\,6\,8)$ and $\rho = (2\,3\,4)$ be permutations in S_{10}. Find the product $\tau\sigma\rho$ and write your answer using cycle notation.
Answer: $\tau\sigma\rho = (3\,5)(2\,4\,6\,8)(2\,3\,4) = (1)(2\,5\,3\,6\,8)(4)(7)(9)(10) = (2\,5\,3\,6\,8)$.

Remark 1.2.4. Taking products of permutations is not commutative. For example:

- $(1\,2\,3)(1\,5\,3\,4) = (1\,5)(2\,3\,4)$
- $(1\,5\,3\,4)(1\,2\,3) = (1\,2\,4)(3\,5)$

Remark 1.2.5. As we have seen, taking products of cycles can be fiddly. However, taking products of <u>disjoint</u> cycles is easy to calculate! To see why, suppose the following are disjoint cycles:

$$(a_1\,a_2\,\ldots\,a_\alpha),(b_1\,b_2\,\ldots\,b_\beta),\ldots,(x_1\,x_2\,\ldots\,x_\xi)$$

Their product is:

$$\begin{aligned} a_1 \to a_2 \to \cdots \to a_\alpha \to a_1 &\quad \text{and} \\ b_1 \to b_2 \to \cdots \to b_\beta \to b_1 &\quad \text{and} \\ \cdots &\quad \text{and} \\ x_1 \to x_2 \to \cdots \to x_\xi \to x_1 & \end{aligned}$$

In cycle notation this is just:

$$(a_1\,a_2\,\ldots\,a_\alpha)(b_1\,b_2\,\ldots\,b_\alpha)\cdots(x_1\,x_2\,\ldots\,x_\xi)$$

Example 1.2.6. Let $\sigma = (1\,2\,3)$ and $\rho = (4\,7\,5)$ be permutations in S_8. Then $\sigma\rho$ maps:

$$1 \to 2 \to 3 \to 1 \quad \text{and} \quad 4 \to 7 \to 5 \to 4 \quad \text{with 6 and 8 fixed}$$

Hence $\sigma\rho = (1\,2\,3)(4\,7\,5)$.

Proposition 1.2.7. *If σ and ρ are <u>disjoint</u> cycles then $\sigma\rho = \rho\sigma$. If σ and ρ are <u>not</u> disjoint cycles then it can happen that $\sigma\rho \neq \rho\sigma$.*

What is the point?!. This result can speed up calculations involving disjoint cycles so remember it!

1.2.1 Handout for Section 1.2

Question 1.2.8. A question to get you thinking about what you've seen so far.

(i) Write down all the equivalent ways of writing down the cycle $(1\,2\,3\,4\,5)$
Answer: $(1\,2\,3\,4\,5), (2\,3\,4\,5\,1), (3\,4\,5\,1\,2), (4\,5\,1\,2\,3), (5\,1\,2\,3\,4)$.

Proof of Proposition 1.2.7. Suppose $\sigma = (a_1\,a_2\,\ldots a_n)$ and $\rho = (b_1\,b_2\,\ldots b_m)$ with $\{a_1,\ldots,a_n\} \cap \{b_1,\ldots,b_m\} = \varnothing$ (i.e. that they are dijoint). Then $\sigma\rho$ is the permutation,

$$\begin{array}{cccccccc} a_1 & a_2 & \ldots & a_n & b_1 & b_2 & \ldots & b_m \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ a_2 & a_3 & \ldots & a_1 & b_2 & b_3 & \ldots & b_1 \end{array} = \begin{array}{cccccccc} b_1 & b_2 & \ldots & b_m & a_1 & a_2 & \ldots & a_n \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ b_2 & b_3 & \ldots & b_1 & a_2 & a_3 & \ldots & a_1 \end{array}$$

which, in cycle notation, equals $(b_1\,b_2\,\ldots b_m)(a_1\,a_2\,\ldots a_n) = \rho\sigma$.
On the other hand, if σ and ρ are not disjoint cycles, then we have already seen in Remark 1.2.4 that it can happen that $\sigma\rho \neq \rho\sigma$. □