

7.2 The signature function

Definition 7.2.1. (Signature function.) Consider $g \in S_n$ and the variables x_1, x_2, \dots, x_n . We can think of g as a permutation of the variables, with $gx_k = x_{gk}$.

(E.g. If $g = (1\ 2\ 3) \in S_3$ then $gx_1 = x_2$, $gx_2 = x_3$, $gx_3 = x_1$.)

We now define a special polynomial:

$$\Delta = \prod_{i < j} (x_i - x_j) = ((x_1 - x_2) \cdots (x_1 - x_n))((x_2 - x_3) \cdots (x_2 - x_n)) \cdots ((x_{n-1} - x_n))$$

(E.g. In S_3 we have $\Delta = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.)

What do permutations in S_n do to Δ ?

(E.g. In S_3 :

- e sends $\Delta \rightarrow \Delta$
- $(1\ 2)$ sends $\Delta \rightarrow (x_2 - x_1)(x_2 - x_3)(x_1 - x_3) = -\Delta$
- $(1\ 3)$ sends $\Delta \rightarrow (x_3 - x_2)(x_3 - x_1)(x_2 - x_1) = -\Delta$

In general case: every permutation in S_n sends Δ to $\pm\Delta$.

We define a map $\sigma : S_n \rightarrow \{1, -1\}$ called *the signature function on S_n* as follows:

$$\sigma(g) = \begin{cases} 1 & \text{if } g \text{ maps } \Delta \text{ to } \Delta \\ -1 & \text{if } g \text{ maps } \Delta \text{ to } -\Delta. \end{cases}$$

If $\sigma(g) = 1$ we say g is *even*, and if $\sigma(g) = -1$ we say g is *odd*.

What is the point?! The signature function divides S_n into two sets (even and odd). We can use this to define a new group that is one of the most important groups there is.

Proposition 7.2.2. Recall that $\{1, -1\}$ is a group under multiplication. The function $\sigma : S_n \rightarrow \{1, -1\}$ is a homomorphism.

Proof. This is an easy proof. Fix $g, h \in S_n$. Then (because $\sigma(g)$ is just ± 1) we have:

$$(gh)\Delta = g(h\Delta) = g(\sigma(h)\Delta) = \sigma(h)(g\Delta) = \sigma(h)\sigma(g)\Delta = \sigma(g)\sigma(h)\Delta.$$

Hence $\sigma(gh) = \sigma(g)\sigma(h)$. □

Note: all the above is the definition of σ , but it's a horrible way to calculate σ . In fact there is an easy way, but we will have to prove some results first.

Corollary 7.2.3. (Very important!) Let $g \in S_n$; we know already we can write g as a product of cycles, $g = c_1 c_2 \dots c_m$. Suppose cycle c_i has length r_i for $i = 1, \dots, m$. Then

$$\sigma(c_i) = (-1)^{r_i - 1},$$

and therefore,

$$\sigma(g) = \sigma(c_1)\sigma(c_2)\cdots\sigma(c_m) = (-1)^{(r_1-1)}\cdots(-1)^{(r_m-1)}.$$

Example 7.2.4. The signature of $g = (15742)(129)$ is $\sigma(g) = (-1)^{5-1}(-1)^{3-1} = 1$, so g is even.

Proposition 7.2.5. If $g \in S_n$ is a permutation, then $\sigma(g^{-1}) = \sigma(g)$.

Proof. Exercise. [Hint: first think about cycles, see handout for proof]. □

Remark 7.2.6. Notice that the signature of a function depends only on its cycle shape. Two permutations with the same cycle shape have the same signature!

7.2.1 Handout for Section 7.2

Example 7.2.7. Here are some examples.

1. $g = (1\ 7\ 3\ 4\ 2)(9\ 5\ 6)(8\ 11) \in S_{12}$ is odd because $\sigma(g) = (-1)^{5-1}(-1)^{3-1}(-1)^{2-1} = -1$.
2. $h = (1\ 2\ 3) \in S_{12}$ is even because $\sigma(h) = (-1)^{3-1} = 1$.
3. To find out directly if gh is even or odd we first need to write it as a product of disjoint cycles because our result only holds for products of disjoint cycles. Now,

$$gh = (1\ 7\ 3\ 4\ 2)(9\ 5\ 6)(8\ 11)(1\ 2\ 3) = (2\ 4)(3\ 7)(5\ 6\ 9)(8\ 11),$$

so $\sigma(gh) = (-1)^{2-1}(-1)^{2-1}(-1)^{3-1}(-1)^{2-1} = -1$, so gh is odd.

However, we could calculate $\sigma(gh)$ more easily (when we know $\sigma(g)$ and $\sigma(h)$) by remembering that σ is a homomorphism: $\sigma(gh) = \sigma(g)\sigma(h) = (-1)(1) = -1$.

4. Every 3-cycle is even.
5. Any product of even permutations is even.

Proof of Proposition 7.2.5. The signature of a cycle of length r is just $(-1)^{r-1}$. For any cycle c of length r we know that c^{-1} is just c written backwards, so c^{-1} is also a cycle of length r . Hence $\sigma(c^{-1}) = (-1)^r = \sigma(c)$. Now any permutation g can be written as a product of disjoint cycles $g = c_1c_2\cdots c_m$, and we have seen that $g^{-1} = c_1^{-1}c_2^{-1}\cdots c_m^{-1}$. Hence $\sigma(g^{-1}) = \sigma(c_1^{-1})\sigma(c_2^{-1})\cdots\sigma(c_m^{-1}) = \sigma(c_1)\sigma(c_2)\cdots\sigma(c_m) = \sigma(g)$. \square