8.2 More group actions and Cayley's Theorem

Example 8.2.1 (Important example)**.** We are used to thinking of S_5 acting on the set $\{1,2,3,4,5\}$, but it can also act on the following sets:

1. $X = \{1,2,3,4,5,6,7,8\}$ with
$$\lambda(g)x = \begin{cases} gx \text{ if } 1 \leq x \leq 5; \text{ and} \\ x \text{ if } 6 \leq x \leq 8 \end{cases}$$
(Check that you agree this is an action, by verifying that (1) and (2) from Definition 8.1.1 both hold).
2. $X = \mathbb{Z}$. For $x \in X$ write $x = 5n + k$ for some $n \in \mathbb{Z}$ and $1 \leq k \leq 5$, and define,
$$\lambda(g)x = 5n + gk.$$
(Check that you agree this is an action, by verifying that (1) and (2) from Definition 8.1.1 both hold).
3. $X = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$ (all pairs of distinct integers between 1 and 5) with
$$\lambda(g)\{a,b\} = \{ga, gb\}.$$
(Check that you agree this is an action, by verifying that (1) and (2) from Definition 8.1.1 both hold).

Recall that an action λ is a homomorphism, and so we can talk about things like the kernel and image of the action.

Proposition 8.2.2. *Let λ be an action of a group G on a set X. The kernel of λ is those elements in g that fix every $x \in X$,*
$$\mathrm{Ker}(\lambda) = \{\mathrm{g} \in \mathrm{G} : \lambda(\mathrm{g})\mathrm{x} = \mathrm{x} \quad \forall \mathrm{x} \in \mathrm{X}\}.$$

Proof. Note that $\lambda(g)x = x$ for all $x \in X$ holds if and only if $\lambda(g) = e$. But $\lambda(g) = e$ if and only if $g \in \mathrm{Ker}(\lambda)$. □

Theorem 8.2.3. (Cayley's Theorem) *Every group is isomorphic to a permutation group.*

Proof. Let $X = G$, with $\lambda(g)x = gx$ for all $g \in G$ and $x \in X$. We have seen this before in Example 8.1.7 (it is called the regular action) and so we know already this is an action. Let $H = \mathrm{Im}(\lambda)$ and note that $H \leq \mathrm{Sym}(X)$, so H is a permutation group.
Since λ is a homomorphism, we can apply the First Isomorphism Theorem to obtain that
$$G/\mathrm{Ker}(\lambda) \cong \mathrm{Im}(\lambda) = \mathrm{H}.$$

If we can show that $\mathrm{Ker}(\lambda) = \langle \mathrm{e} \rangle$, then (because $G/\langle e \rangle = G$) we will have proven that $G \cong H$ and our proof will be complete.
So, suppose $g \in \mathrm{Ker}(\lambda)$. Then $g \in G$ and $gx = x$ for all $x \in X$. Since $X = G$ we know that X is a group, so we have $gxx^{-1} = xx^{-1}$. Hence $g = e$. Thus, $\mathrm{Ker}(\lambda) = \langle \mathrm{e} \rangle$. □

What is the point?!. We've already seen that every permutation group is a group. Now Cayley's Theorem is telling us that every group can be viewed as a permutation group (called its permutation representation).