8.3 Orbits and stabilisers

Definition 8.3.1. Let X be a G-set. The *orbit* of $x \in X$ is

$$Gx = \{\lambda(g)x : g \in G\}.$$

It is the set of all images of x under the action of G. In the special case that $Gx = X$, we say that the action is *transitive*.

Definition 8.3.2. Fix $x \in X$. The *stabiliser* of x is denoted by $\mathrm{Stab}_G(x)$, and is defined to be

$$\mathrm{Stab}_G(x) = \{g \in G : \lambda(g)x = x\}.$$

Example 8.3.3. $G = S_4$ and $X = \{1, 2, 3, 4\}$. Note X is a G-set via the action $\lambda(g)i = gi$.

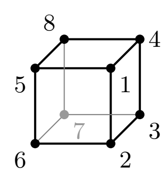
- **Orbits:** $G1 = \{1, 2, 3, 4\} = G2 = G3 = G4$. This action is transitive.
- **Stabilisers:** $\mathrm{Stab}_G(1) = \{\text{all things in } S_4 \text{ that fix } 1\} = \{e, (2\,3\,4), (2\,4\,3), (2\,3), (2\,4), (3\,4)\}$ $\mathrm{Stab}_G(2) =$

  $\{\text{all things in } S_4 \text{ that fix } 2\} = \{e, (1\ 3\ 4), (1\ 4\ 3), (1\ 3), (1\ 4), (3\ 4)\}$

  Etc. In all cases note that $\mathrm{Stab}_G(i) \cong S_3$.

Example 8.3.4. (Easy example). Recall from Example 8.1.9 that there is a natural action of $D_8 = \{e, \rho, \rho^2, \rho^3, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3\}$ (where $\rho = (1\ 2\ 3\ 4)$ and $\sigma = (1\ 4)(2\ 3)$) on the cube $\mathcal{C}$ below, via:

$$\lambda(\rho) = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8) \quad \text{and} \quad \lambda(\sigma) = (1\ 4)(2\ 3)(5\ 8)(6\ 7) \quad \text{and} \quad \lambda(\sigma^i\rho^j) = \lambda(\sigma)^i\lambda(\rho)^j.$$



- **Orbits:** The orbit of corner 1 is $\{1, 2, 3, 4\}$ (this is also the orbit of corner 2, corner 3 and corner 4) and the orbit of corner 5 is $\{5, 6, 7, 8\}$. The action is not transitive.
- **Stabilisers:** For corner 1 we have that $\mathrm{Stab}_{D_8}(1) = \{e, (2\ 4)(6\ 8)\} = \{e, \sigma\rho^3\}$. The stabilisers of other corners can be worked out similarly.

Example 8.3.5. Recall from Example 8.1.7 that any group G acts on itself (i.e. $X = G$) via the regular action: $\lambda(g)x = gx$ for all $g \in G$ and all $x \in X = G$.

- **Orbits:** The orbit of $e \in X$ is $Ge = \{ge : g \in G\} = G = X$. The action is transitive.

  [Exercise: show that the orbit of any $x \in X$ is also equal to X.]
- **Stabilisers:** For any $x \in X = G$ notice that $gx = x \iff g = e$. Therefore we have that $\mathrm{Stab}_G(x) = \{g \in G : \lambda(g)x = x\} = \langle e \rangle$.

Example 8.3.6. Recall from Example 8.1.8 that any group G acts on itself (i.e. $X = G$) via the conjugation action: $\lambda(g)x = gxg^{-1}$ for all $g \in G$ and all $x \in X = G$.

- **Orbits:** The orbit of $e \in X$ is $Ge = \{geg^{-1} : g \in G\} = \{e\}$. The action is not transitive when $G \neq \langle e \rangle$. Working out $Gx = \{gxg^{-1} : g \in G\}$ for other elements $x \in G$ depends on the group G.

  [Exercise: if G is abelian show that the orbit of any $x \in X$ is equal to $\{x\}$.]
- **Stabilisers:** For any $x \in X = G$, by definition $\mathrm{Stab}_G(x) = \{g \in G : gxg^{-1} = x\} = \{g \in G : g^{-1}xg = x\}$. This set plays a special role in group theory and is called the *centraliser of x in G*.

8.3.1 Handout for Section 8.3

Example 8.3.7. $H = \{e, (12)(34)\}$ and $X = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$. Note X is an H-set via the action $\lambda(h)\{i,j\} = \{hi, hj\}$.

- **Orbits:** $H\{1,2\} = \{\{1,2\}\}$ $H\{1,3\} = \{\{1,3\},\{2,4\}\}$ H$\{1, 4\} = \{\{1, 4\}, \{2, 3\}\}$

  $H\{2,3\} = \{\{2,3\},\{1,4\}\}$ $H\{2,4\} = \{\{2,4\},\{1,3\}\}$ H$\{3,4\} = \{\{3,4\}\}$

  Notice that $H\{1,3\} = H\{2,4\}$ and $H\{1,4\} = H\{2,3\}$. This action is not transitive.
- **Stabilisers:** $\mathrm{Stab}_H(\{1,2\}) = \{$all things in H that either fix 1 and 2 or interchange 1 and 2$\}$

  So $\mathrm{Stab}_H(\{1,2\}) = \{e, (1\ 2)(3\ 4)\}$ and similarly $\mathrm{Stab}_H(\{3,4\}) = \{e, (1\ 2)(3\ 4)\}$.

  On the other hand,

  $$\mathrm{Stab}_H(\{1,3\}) = \{e\}, \quad \mathrm{Stab}_H(\{2,4\}) = \{e\}, \quad \mathrm{Stab}_H(\{1,4\}) = \{e\}, \quad \mathrm{Stab}_H(\{2,3\}) = \{e\}$$

Question 8.3.8. Let $K = \{e, (123), (132)\}$ and $X = \{1,2,3,4\}$. Note X is a K-set via the action $\lambda(g)i = gi$. Find the orbits of K on X and the stabilisers of all things in X. Is the action transitive?

- **Orbits:** $K1 = \{1,2,3\} = K2 = K3$ and $K4 = \{4\}$. The action is not transitive.
- **Stabilisers:** $\mathrm{Stab}_K(1) = \mathrm{Stab}_K(2) = \mathrm{Stab}_K(3) = e$ and $\mathrm{Stab}_K(4) = K$.

9 The orbit stabiliser theorem and the orbit counting theorem

9.1 The orbit stabiliser theorem

Here we note some facts that you will prove on problems sheets.

Proposition 9.1.0. *Let X be a G-set and $x, y \in X$.*

(i) Either $Gx = Gy$ or $Gx \cap Gy = \varnothing$. In other words, orbits are disjoint or they are equal.

(The above result means that every G-set can be partitioned into a union of disjoint orbits $X = \mathrm{Orb}_1 \cup \mathrm{Orb}_2 \cup \ldots \cup \mathrm{Orb}_n$.)

(ii) If x and y lie in the same orbit, then $Gx = Gy$.

(iii) If Y is an orbit and $x \in Y$, then $Gx = Y$.

(iv) $\mathrm{Stab}_G(x)$ is a subgroup of G.

If you find a partition of X into orbits $\mathrm{Orb}_1 \cup \ldots \cup \mathrm{Orb}_n$ and we choose an element $x_i \in \mathrm{Orb}_i$ from each orbit, then $\{x_1, \ldots, x_n\}$ is called a *set of orbit representatives* of the G-set X.

We now give the Orbit Stabiliser Theorem. This powerful result allows us to calculate the size of any orbit once we know the size of the appropriate stabiliser. We can use this to answer questions like: "Can D_{10} act transitively on the corners of a triangle?"

Theorem 9.1.1. (Orbit Stabiliser Theorem.) *Let G be a group, and let X be a G-set. For all $x \in X$ the size of the orbit Gx is given by the formula,*

$$|Gx| = [G : \mathrm{Stab}_G(x)].$$

In particular, if G is finite then we have $|Gx| = |G|/|\mathrm{Stab}_G(x)|$.

Proof. To simplify our notation, we define $S = \mathrm{Stab}_G(x)$. Let λ be the action of G on X. Fix $g, h \in G$ and note that $\lambda(g)x, \lambda(h)x \in Gx$. Now

$$\begin{aligned} \lambda(g)x \neq \lambda(h)x &\iff \lambda(h^{-1})\lambda(g)x \neq \lambda(h^{-1})\lambda(h)x \\ &\iff \lambda(h^{-1}g)x \neq \lambda(h^{-1}h)x \\ &\iff \lambda(h^{-1}g)x \neq x \\ &\iff h^{-1}g \notin S \\ &\iff gS \neq hS \qquad \text{(by Theorem 4.1.3)} \end{aligned}$$

Since $\lambda(g)x \neq \lambda(h)x \iff gS \neq hS$, the number of distinct elements in the orbit Gx is equal to the number of distinct cosets of S in G, which we recall is denoted by $[G : \mathrm{Stab}_G(x)]$. This proves the main part of the theorem.

Finally, we notice that if G is finite, then by Lagrange's Theorem we have $[G : \mathrm{Stab}_G(x)] = |G|/|\mathrm{Stab}_G(x)|$. □

Example 9.1.2. Recall that in Example 8.3.3 we had $G = S_4$ and $X = \{1, 2, 3, 4\}$, with X a G-set via the action $\lambda(g)i = gi$. In that example we saw that $G1 = \{1, 2, 3, 4\}$ so $|G1| = 4$. We also saw that $\mathrm{Stab}_G(1) = \{e, (2\ 3\ 4), (2\ 4\ 3), (2\ 3), (2\ 4), (3\ 4)\}$, so $|\mathrm{Stab}_G(1)| = 6$. The Orbit-Stabiliser Theorem tells us that $|G1| = |G|/|\mathrm{Stab}_G(1)|$, which is true in this case, since $4 = 4!/6$.

Example 9.1.3. Let G be a group of order 20. We can prove, using the Orbit Stabiliser Theorem, that there is no transitive action of G on a set of size 6.

First, let's recall the definition of a transitive action. An action λ of G on a set X is transitive if there is some $x \in X$ such that the orbit Gx is all of X.

Suppose X has size 6 and such a transitive action λ exists. Then there is some $x \in X$ such that $Gx = X$. Hence $|Gx| = 6$. Using the Orbit-Stabiliser Theorem, we have

$$6 = |Gx| = |G|/|\mathrm{Stab}_G(x)| = 20/|\mathrm{Stab}_G(x)|.$$

Since $|\mathrm{Stab}_G(x)|$ is an integer, this is impossible. Hence there is no orbit of G of size 6.

9.1.1 Handout for Section 9.1

Question 9.1.4. Can D_{10} act transitively on the corners of a triange?

Solution: No! Suppose the answer is yes. Then D_{10} has a transitive action on a set X of size 3. Hence there is some $x \in X$ such that $|D_{10}x| = 3$. By the Orbit-Stabiliser Theorem we have $3 = |D_{10}x| = |D_{10}|/|\mathrm{Stab}_{D_{10}}(x)| = 10/|\mathrm{Stab}_{D_{10}}(x)|$, which is impossible because $|\mathrm{Stab}_{D_{10}}(x)|$ is an integer.

Example 9.1.5. Let $G = S_n$ and let $Y = \{1, 2, \ldots, n\}$. There is a natural action of G on the set $Y \times Y$ via the action,

$$\lambda(g)(x, y) = (gx, gy).$$

So for example $\lambda((1\ 2\ 3))(1, 3) = ((1\ 2\ 3)1, (1\ 2\ 3)3) = (2, 3)$ and $\lambda((1\ 2))(2, 3) = ((1\ 2)2, (1\ 2)3) = (1, 3)$.
What is the size of the orbit $G(1, 1)$? We could calculate this directly, or we could use the Orbit-Stabiliser Theorem. To use the theorem, we need to know $|G| = n!$ and $|\mathrm{Stab}_G((1, 1))|$.
Now $\mathrm{Stab}_G((1, 1)) = \{\text{all things in } S_n \text{ that fix } 1\} = \mathrm{Sym}(\{2, 3, \ldots, n\})$. We know that $\mathrm{Sym}(\{2, 3, \ldots, n\})$, being the group of all permutations of $n - 1$ things, has size $(n - 1)!$. Hence $|\mathrm{Stab}_G((1, 1))| = (n - 1)!$.
Now we can use the Orbit Stabiliser Theorem to calculate that $|G(1, 1)| = |G|/|\mathrm{Stab}_G((1, 1))| = n!/(n-1)! = n$.