9.2 The orbit counting theorem

Here is a question: suppose you have lots of cubes, and you want to colour their faces using three colours. How many different cubes are possible?
The difficulty in the question is that you might think you've coloured the cubes differently, but then when you pick them up and rotate them, some will be the same.
To answer this question, we need to know about the orbits of the group of rotations of a cube. We need a way of counting them.

The following result is sometimes called Burnside's Lemma, but it's also sometimes called Not Burnside's Lemma. In fact, it is due to Cauchy and Frobenius. Burnside just wrote about it in his book.

Definition 9.2.1. Suppose Y is a G-set with action λ, and $g \in G$. Then $\mathrm{Fix_Y(g) = \{y \in Y : \lambda(g)y = y\}}$ is the set of points in Y that are fixed by g.

Don't confuse $\mathrm{Fix_Y(g)}$ and $\mathrm{Stab}_G(y)$. The first is all things in Y that are fixed by g; the second is all things in G that fix y.

Example 9.2.2. We know that S_5 acts on $Y = \{1,2,3,4,5\}$. If $\tau = (1\ 2\ 5)$ and $\rho = (1\ 2\ 3\ 4\ 5)$ are permutations in S_5, then $\mathrm{Fix_Y}(\tau) = \{3,4\}$ and $\mathrm{Fix_Y}(\rho) = \varnothing$ and $\mathrm{Fix_Y(e) = Y}$.

Theorem 9.2.3. (Orbit Counting Theorem.) *Let G be a finite group acting on a set Y. Then:*

$$\textit{No. of orbits of } G \textit{ on } Y = \frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix_Y(g)}|$$

In other words, the number of orbits is equal to the average number of fixed points.

Proof. Let $\mathcal{P} \subseteq Y \times G$ be the set $\mathcal{P} = \{(y,g) : \lambda(g)y = y\}$. We count $|\mathcal{P}|$ in two different ways.
Firstly, note that we can write $\mathcal{P}$ as $\{(y,g) : g \in G,\ y \in \mathrm{Fix_Y(g)}\}$, and so

$$|\mathcal{P}| = \sum_{g\in G}|\mathrm{Fix_Y(g)}|. \tag{1}$$

Secondly, note that we can write $\mathcal{P}$ as $\{(y,g) : y \in Y,\ g \in \mathrm{Stab}_G(y)\}$ and so (by the Orbit-Stabiliser Theorem),

$$|\mathcal{P}| = \sum_{y\in Y}|\mathrm{Stab}_G(y)| = \sum_{y\in Y}\frac{|G|}{|Gy|}.$$

Now Y can be broken down as a union of disjoint orbits, $Y = \mathrm{Orb_1} \cup \ldots \cup \mathrm{Orb_m}$ and for any two $y, y' \in \mathrm{Orb_i}$ we have $Gy = Gy' = \mathrm{Orb_i}$. We can therefore rewrite our sum as,

$$|\mathcal{P}| = \sum_{y\in Y}\frac{|G|}{|Gy|} = \sum_{i=1}^{m}\left(\sum_{y\in \mathrm{Orb_i}}\frac{|G|}{|Gy|}\right) = \sum_{i=1}^{m}\left(|\mathrm{Orb_i}|\cdot\left(\frac{|G|}{|\mathrm{Orb_i}|}\right)\right) = m|G|. \tag{2}$$

Combining equations 1 and 2 we get,

$$m|G| = \sum_{g\in G}|\mathrm{Fix_Y(g)}|.$$

Since m is the number of orbits of G on Y, the theorem follows immediately. □

Example 9.2.4. Consider the group $G = \{e, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}$. It has an obvious action on the set $Y = \{1,2,3,4\}$ given by $\lambda(g)y = gy$ for all $g \in G$ and all $y \in Y$.
Let's calculate the number of orbits of G on Y using the Orbit Counting Theorem.

$$\mathrm{Fix_Y(e) = Y}, \quad \mathrm{Fix_Y}((1\ 2)) = \{3,4\}, \quad \mathrm{Fix_Y}((3\ 4)) = \{1,2\}, \quad \mathrm{Fix_Y}((1\ 2)(3\ 4)) = \varnothing.$$

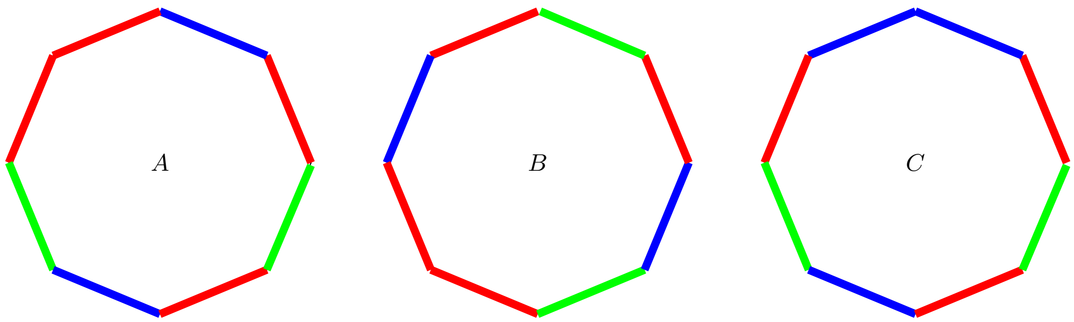
Hence the number of orbits of G on Y is $m = \frac{1}{4}(4+2+2+0) = 2$. What are these orbits? By inspection we see the two orbits are $\{1,2\}$ and $\{3,4\}$.

This example is trivial. Our next example is something harder, that will help us with the cube question.

9.3 An example: colouring octagons

Our next example will help us with the cube question.

Example 9.3.1. Suppose we colour the edges a regular octagon using three colours. How many different coloured octagons can we obtain? In the diagram below we have three such octagons, A, B, C. Notice that A and B are the same (one is just the other rotated) while C is different (it has more blue edges). Problems like this occur frequently in things like the development of pharmaceuticals.

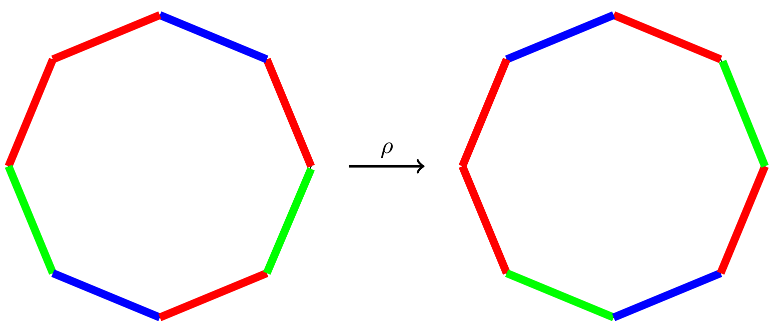


Colouring the edges of three octagons using three colours

Two coloured octagons are indistinguishable from each other if and only if we can move from one to the other using a symmetry of an octagon — that is if D_{16} can transform one coloured octagon into the other.

Time to use group actions in a fancy way! Let $\mathcal{O}$ be an octagon whose edges are coloured using colours $\{r, b, g\}$. Let X be the set of all possible ways of edge-colouring $\mathcal{O}$ (starting from the top going anticlockwise), so for example A above gives us $(r, r, g, b, r, g, r, b) \in X$ and B gives us $(r, b, r, r, g, b, r, g) \in X$ and C gives us $(b, r, g, b, r, g, r, b) \in X$. Clearly $|X| = 3^8$.

Now D_{16} acts on X in the obvious way. For example (see digram below), the usual anticlockwise rotation $\rho = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8) \in D_{16}$ acts on X by sending (r, r, g, b, r, g, r, b) to (b, r, r, g, b, r, g, r).



Now we see that two coloured octagons are the same if and only if they lie in the same orbit of this action of D_{16} on X. Therefore the number of distinct ways of edge colouring an octagon using three colours is equal to the number of orbits of D_{16} on X.
We calculate this using the Orbit Counting Lemma. For this we calculate $|\mathrm{Fix_X(g)}|$ for each $g \in D_{16}$. Of course we have,

$$\mathrm{Fix_X(e) = X} \quad \text{so} \quad |\mathrm{Fix_X(e)}| = 3^8.$$

Now, ρ only fixes elements of X that lie in $\{(a, a, a, a, a, a, a, a) : a \in \{r, b, g\}\}$, so,

$$|\mathrm{Fix_X}(\rho)| = 3.$$

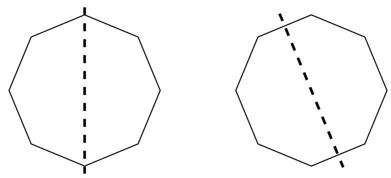
On the other hand, ρ^2 fixes elements of X that lie in $\{(a,c,a,c,a,c,a,c) : a,c \in \{r,b,g\}\}$, so

$$|\mathrm{Fix}_X(\rho^2)| = 3^2.$$

Continuing in this way you can work out that,

$$|\mathrm{Fix}_X(\rho^3)| = 3, \quad |\mathrm{Fix}_X(\rho^4)| = 3^4, \quad |\mathrm{Fix}_X(\rho^5)| = 3,$$
$$|\mathrm{Fix}_X(\rho^6)| = 3^2, \quad |\mathrm{Fix}_X(\rho^7)| = 3$$

Now let's turn to the reflections. We can name the reflections any way we want, so let's take $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ to be the four reflections whose lines of reflection are through two corners, and let $\sigma_5, \sigma_6, \sigma_7, \sigma_8$ be the four reflections whose lines of reflection are through two edges.



Two different types of reflection of $\mathcal{O}$: corner-corner or edge-edge

One of these reflections (through two corners) is $\sigma_1 = (2\ 8)(3\ 7)(4\ 6)$. It fixes elements of X that lie in $\{(a,c,d,e,e,d,c,a) : a,c,d,e \in \{r,b,g\}\}$. Hence $|\mathrm{Fix}_X(\sigma_1)| = 3^4$. Reflections σ_2, σ_3, and σ_4 are similar. Hence,

$$|\mathrm{Fix}_X(\sigma_1)| = |\mathrm{Fix}_X(\sigma_2)| = |\mathrm{Fix}_X(\sigma_3)| = |\mathrm{Fix}_X(\sigma_4)| = 3^4.$$

Another one of these reflections (through one edge) is $\sigma_5 = (1\ 2)(3\ 8)(4\ 7)(5\ 6)$. It fixes elements of X that lie in $\{(a,c,d,e,f,e,d,c) : a,c,d,e,f \in \{r,b,g\}\}$. Hence $|\mathrm{Fix}_X(\sigma_5)| = 3^5$. Reflections σ_6, σ_7, and σ_8 are similar. Hence,

$$|\mathrm{Fix}_X(\sigma_5)| = |\mathrm{Fix}_X(\sigma_6)| = |\mathrm{Fix}_X(\sigma_7)| = |\mathrm{Fix}_X(\sigma_8)| = 3^5.$$

Plugging all this into the Orbit Counting Theorem we have,

$$m = \frac{1}{16}(3^8 + 3 + 3^2 + 3 + 3^4 + 3 + 3^2 + 3 + 4 \cdot 3^4 + 4 \cdot 3^5) = \frac{7968}{16} = 498.$$

9.4 Solving the cube question

Example 9.4.1. If you colour the faces of a cube using three colours, how many different coloured cubes can you obtain?

Try to work this out as an exercise, but be careful here: a cube is 3D in 3D space, whereas the octagon in Example 9.3.1 was 2D in 3D space. We could reflect the octagon (by "flipping" it through the 3rd dimension). However, we can't do this with a cube because it is 3D — a reflected 3D object is fundamentally different from the original object (e.g. your right and left hands).
After you've tried this exercise, an outline of the solution is given below. (Hint: make sure you only consider rotational symmetries).

Outline solution. We described the full symmetry group of a cube $\mathcal{C}$ in Example 3.3.2. However, as mentioned above, two coloured cubes are only the same if you can get from one to the other using rotations (since reflections change the object). Therefore the group we are interested in is $\mathrm{Rot}(\mathcal{C})$.
We described the group $\mathrm{Rot}(\mathcal{C})$ of rotations of a cube $\mathcal{C}$ in Example 3.3.2. Look back at this example, and remind yourself we have:

- One identity rotation — type (A)
- Six 90 degree face rotations — type (B)
- Three 180 degree face rotations — type (C)
- Eight 120 corner rotations — type (D)
- Six 180 edge rotations — type (E)

So 24 symmetries. Now (as in Example 3.3.2) we let X be the set of all possible ways of face-colouring $\mathcal{C}$, and note that $\mathrm{Rot}(\mathcal{C})$ acts on X in the obvious way. We have that $|X| = 3^6$ because there are 3 choices for the colour on each face.
Now think about the face-colourings of $\mathcal{C}$ that are fixed by the elements of $\mathrm{Rot}(\mathcal{C})$ (it helps here to look back at the diagrams in Example 3.3.2).

- The type (A) rotation g fixes all possible face-colourings, so $|\mathrm{Fix_X(g)}| = 3^6$.
- Each type (B) rotation g fixes all face-colourings in which the top and bottom faces receive any colour and the sides all receive the same colour, so $|\mathrm{Fix_X(g)}| = 3^3$.
- Each type (C) rotation g satisfies $|\mathrm{Fix_X(g)}| = 3^4$
- Each type (D) rotation g satisfies $|\mathrm{Fix_X(g)}| = 3^2$
- Each type (E) rotation g satisfies $|\mathrm{Fix_X(g)}| = 3^3$.

Plugging all this into the Orbit Counting Theorem we have that the number of orbits m is,

$$m = \frac{1}{24}(3^6 + 6 \cdot 3^3 + 3 \cdot 3^4 + 8 \cdot 3^2 + 6 \cdot 3^3) = 57.$$

Hence the number of different cubes you can obtain by colouring the faces of a cube with three colours is 57.