11 Abelian groups

In this section we will discover the structure of all finite abelian groups.

11.1 Direct products

Recall the definition of a direct product (Definition 2.1.3): If G and H are groups, then their direct product $G \times H$ is also a group, where $G \times H$ is the set $\{(g,h) \mid g \in G, h \in H\}$ with operation

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

This is usually referred to as the *external direct product*. External direct products are easy to work with, but hard to detect. We will now develop a technique for detecting direct products.
Note that for finite groups, $|G \times H| = |G| \cdot |H|$ because we have $|G|$ choices for the first coordinate and $|H|$ choices for the second.

Proposition 11.1.1. *Let G and H be groups. Then $G \times H \cong H \times G$.*

Proof. You will prove this on a problem sheet. □

Lemma 11.1.2. *Suppose M and N are normal subgroups of a group G, and $M \cap N = \langle e_G \rangle$. Then for all $m \in M$ and all $n \in N$ we have $mn = nm$.*

Proof. Note that $n^{-1}mn \in M$ (because $M \trianglelefteq G$) and so $n^{-1}mnm^{-1} \in M$. Also $mnm^{-1} \in N$ (because $N \trianglelefteq G$) and so $n^{-1}mnm^{-1} \in N$. Hence $n^{-1}mnm^{-1} \in M \cap N = \langle e_G \rangle$. Therefore $n^{-1}mnm^{-1} = e_G$ and so $mn = nm$. □

Theorem 11.1.3. *Suppose M and N are normal subgroups of a group G, and $M \cap N = \langle e_G \rangle$. Then $MN \leq G$ and*

$$MN \cong M \times N.$$

Proof. By Lemma 4.2.4, we know that $MN \leq G$. Every element in $g \in MN$ can be written $g = mn$, for some $m \in M$ and some $n \in N$. In fact, this way of writing g is unique, because if $g = mn = m_1 n_1$ for some $m_1 \in M$ and $n_1 \in N$ then $nn_1^{-1} = m^{-1}m_1$. Hence both $m^{-1}m$ and nn_1^{-1} lie in $M \cap N = \langle e_G \rangle$. Therefore $n = n_1$ and $m = m_1$.
Now define a function $\theta : MN \to M \times N$ with $\theta(mn) = (m,n)$. By the above argument, θ is well-defined.

- θ is a homomorphism: if $m_1, m_2 \in M$ and $n_1, n_2 \in N$, then $\theta(m_1 n_1)\theta(m_2 n_2) = (m_1, n_1)(m_2, n_2) = (m_1 m_2, n_1 n_2) = \theta(m_1 m_2 n_1 n_2)$, and by Lemma 11.1.2 we have that $m_1 m_2 n_1 n_2 = m_1 n_1 m_2 n_2$.
- θ is one-to-one: if $\theta(m_1 n_1) = \theta(m_2 n_2)$ then $(m_1, n_1) = (m_2, n_2)$ and therefore $m_1 = m_2$ and $n_1 = n_2$.
- θ is onto: any $(m,n) \in M \times N$ satisfies $\theta(mn) = (m,n)$.

□

Remark 11.1.4. The converse to Theorem 11.1.3 is obviously true: the group $M \times N$ has normal subgroups $\hat{M} = M \times \langle e_N \rangle$ and $\hat{N} = \langle e_M \rangle \times N$ and it is easy to check that $\hat{M}\hat{N} = M \times N$ and $\hat{M} \cap \hat{N} = \langle (e_M, e_N) \rangle = \langle e_{M \times N} \rangle$.

Definition 11.1.5. If G is a group with normal subgroups M and N such that $M \cap N = \langle e_G \rangle$ and $G = MN$, then G is said to be the *internal direct product* of M and N.

What is the point?!. Internal and external direct products are essentially the same thing. They each have advantages: external direct products are easy to work with and internal direct products are easy to detect (just look for two normal subgroups whose intersection is trivial).