2.4 Groups generated by sets

Recall that the notation $\langle g \rangle$ for some $g \in G$ means the group generated by g. It is the smallest subgroup of G that contains g. (You might like to try to prove this!) In particular,

$$\langle g \rangle = \{g^n : n \in \mathbb{Z}\}.$$

We can extend this definition to include more generators. For example, if $g_1, g_2 \in G$ we can define $\langle g_1, g_2 \rangle$ to be all possible powers of multiples of g_1 and g_2, like this:

$$\langle g_1, g_2 \rangle = \{g_1^{n_1} g_2^{k_1} g_1^{n_2} g_2^{k_2} \cdots g_1^{n_m} g_2^{k_m} : m \in \mathbb{N} \text{ and } n_i, k_i \in \mathbb{Z} \text{ for all } 1 \le i \le m\}$$

It is easy to see that this is a group. But why stop at only two elements?!

Definition 2.4.1. Let G be a group and let $S \subseteq G$. Then the *group generated by* S is

$$\langle S \rangle = \{a_1^{n_1} a_2^{n_2} \cdots a_m^{n_m} : m \in \mathbb{N} \text{ and } n_i \in \mathbb{Z} \text{ and } a_i \in S \text{ for all } 1 \le i \le m\}.$$

It is the smallest subgroup of G that contains all the elements in S. In particular, $\langle S \rangle \le G$.

A group G is *finitely generated* if there exists a finite subset $S \subseteq G$ for which $G = \langle S \rangle$.

We now have a way of finding loads of subgroups of a group—just take any subset S of the group and look at the subgroup $\langle S \rangle$. (This might sometimes equal the whole group.)

Example 2.4.2. The Klein four-group is a subgroup of S_4. It is the smallest non-cyclic group. It is denoted by K_4, and is defined to be:

$$K_4 = \langle (1\ 2), (3\ 4) \rangle$$

As you can see, it is generated by the two 2-cycles $(1\ 2)$ and $(3\ 4)$. Written out in full, we have:

$$K_4 = \{e, (1\ 2), (3\ 4), (1\ 2)(3\ 4)\}.$$

For a fun exercise, try to draw a shape $\mathcal{S}$ whose symmetry group is K_4.