3 Some groups arising from geometry

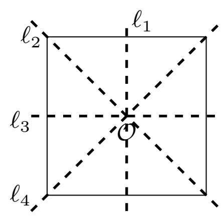
3.1 The dihedral groups part 1: D_8

Recall that a *regular n-gon* is a shape with n-sides whose sides are all the same length, and whose internal angles are all equal. Examples: an equilateral triangle (regular 3-gon); a square (regular 4-gon); a pentagon (regular 5-gon); etc. The symmetries of these shapes form an important class of groups called the *dihedral groups*.
We will look at the a regular 4-gon (square) and will generalise to any regular n-gon.

Imagine we draw a square in the Euclidean plane with centre the origin $\mathcal{O}$. Think of the operations we can perform without changing the square. We only have:

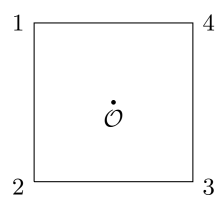
- 4 reflections (reflect through lines $\ell_1, \ell_2, \ell_3, \ell_4$ of reflectional symmetry): $\sigma_1, \sigma_2, \sigma_3, \sigma_4$
- 4 anticlockwise rotations (rotate about the origin with angles $0, \frac{\pi}{2}, \pi$ or $\frac{3\pi}{2}$): $\rho_0, \rho_{\frac{\pi}{2}}, \rho_\pi, \rho_{\frac{3\pi}{2}}$

Anything else will either move the square or distort it so it is no longer a square. These 4 reflections and 4 rotations are called the *symmetries* of the square.



Lines of symmetry

These operations send corners to corners, so they are permuting the corners. If we label the corners $\{1, 2, 3, 4\}$ it is easier to see what is happening.



Square with labelled corners

Rotations: we see that ρ_0 fixes all corners while,

$$\rho_0 = e, \quad \rho_{\frac{\pi}{2}} = (1\,2\,3\,4), \quad \rho_\pi = (1\,3)(2\,4), \quad \rho_{\frac{3\pi}{2}} = (1\,4\,3\,2).$$

Reflections: here we have,

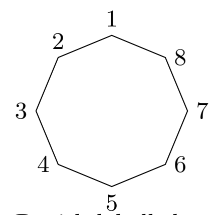
$$\sigma_1 = (1\,4)(2\,3), \quad \sigma_2 = (2\,4), \quad \sigma_3 = (1\,2)(3\,4), \quad \sigma_4 = (1\,3).$$

Notice that the product of any two symmetries gives another symmetry—Of course! If the first operation doesn't change the shape, and nor does the second, then their product won't change the shape either!

The symmetries: $\{\rho_0, \rho_{\frac{\pi}{2}}, \rho_\pi, \rho_{\frac{3\pi}{2}}, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ under the operation of combining symmetries form a group, called *the symmetry group of the square* (also: *the isometry group of the square*). It also has another name: the *dihedral group of order* 8. Because the dihedral group of order 8 permutes the four corners, we can think of it as a subgroup of S_4. Note it is called the Dihedral group of order 8 because it contains 8 permutations — the order of the group is 8.

3.2 The dihedral groups part 2: D_{2n}

Now let's consider a regular n-gon $\mathcal{R}$ for some $n \geq 3$ (again drawn in the Euclidean plane with the origin at the centre). We label its corners anticlockwise in order: $1, 2, \ldots, n$.



Regular n-gon $\mathcal{R}$ with labelled corners, for $n = 8$

- Observation 1: $\mathcal{R}$ has at most $2n$ symmetries.

  Proof: Any symmetry τ of a regular n-gon must map corners to corners, and preserve distance (so adjacent corners are mapped to adjacent corners).

  How many choices do we have for τ? Once the image of corner 1 under τ has been decided (n choices for this), we know that the image of corner 2 must be adjacent to the image of corner 1 (and there are just 2 choices for this) and after this the images of all other corners are determined (so no more choice!). Hence there are $2n$ choices, therefore there are at most $2n$ symmetries of our regular n-gon.
- Observation 2: $\mathcal{R}$ has at least n lines of reflectional symmetry.

  Proof: There is one line of reflectional symmetry through each vertex, and one line of reflectional symmetry passing through the midpoint of each edge. This gives $2n$ lines. However, we have double-counted, since each line passes through the boundary of $\mathcal{R}$ twice. Hence n lines.
- Observation 3: $\mathcal{R}$ has at least n rotational symmetries.

  Proof: An anticlockwise rotation ρ about the origin by $\frac{2\pi}{n}$ permutes the corners as the permutation $(1\,2\,\ldots\,n)$. This permutation has order n so $\{\rho^0, \rho^1, \ldots, \rho^{n-1}\}$ are all distinct.
- Observation 4: No rotational symmetry of $\mathcal{R}$ is a reflectional symmetry, and no reflectional symmetry is a rotational symmetry.

  Proof: A rotation preserves the order of the corner labelling (counterclockwise as $1, 2, \ldots, n$) while a reflection never preserves this order.

Proposition 3.2.1. *The number of symmetries of a regular n-gon is $2n$.*

Proof. Exercise: see handout. Hint: Just use the observations. □

In fact, we can always think of the symmetries of a regular n-gon as a permutation of its corners in this way.

Definition 3.2.2. The symmetry group of a regular n-gon with corners labelled (anticlockwise) as $1, 2, \ldots, n$ has $2n$ elements and is written

$$D_{2n} = \{e, \rho, \rho^2, \ldots, \rho^{n-1}, \sigma_1, \sigma_2, \ldots, \sigma_n\},$$

where (as a permutation of the corners) $\rho = (1\,2\,\ldots\,n)$ and $\sigma_1, \ldots, \sigma_n$ are reflections through its n lines of reflectional symmetry. This group is called the *dihedral group of order* $2n$. It is a subgroup of S_n.

Why is this a group? It is a subset of S_n, so we need only check it is a subgroup by applying the Quick Subgroup Test.

- $e \in D_{2n}$
- Applying any two symmetries must also give a symmetry, so D_{2n} is closed
- Every element has an inverse:
  - (i) $\sigma_i^{-1} = \sigma_i$ because it is a reflection
  - (ii) $\rho^n = e$, therefore $\rho^{-i} = \rho^{n-i}$

Warning! Some mathematicians (and books) use D_n instead of D_{2n} to refer to the group $\{e, \rho, \rho^2, \ldots, \rho^{n-1}, \sigma_1, \ldots, \sigma_n\}$. Why?! No-one knows. Most mathematicians and textbooks use D_{2n}, but just be careful.

3.2.1 Handout for Section 3.2

Proof of Proposition 3.2.1. Suppose we have a regular n-gon $\mathcal{R}$. By our above observations, we know there are at least n lines of reflectional symmetry and so there are at least n distinct reflections: $\sigma_1, \ldots, \sigma_n$. Each reflection has order 2.

We have also observed above that $\mathcal{R}$ has at least n distinct rotational symmetries, ρ^0, ρ^1, $\ldots, \rho^{n-1}$ and that the set of rotations and the set of reflections is disjoint.

Hence the set $\{e, \rho, \rho^2, \ldots, \rho^{n-1}, \sigma_1, \ldots, \sigma_n\}$ contains $2n$ distinct symmetries of our regular n-gon $\mathcal{R}$. However, we also observed above that $\mathcal{R}$ has at most $2n$ symmetries. Therefore this list must consist of all of them. $\square$