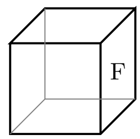
3.3 Symmetry groups of more general shapes

Definition 3.3.1. In fact, if we are given any shape Π in $\mathbb{R}^n$, its symmetries (distance-preserving maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ that send Π to Π) form a group called the *symmetry group of* Π, which we denote in this course by $\text{Isom}(\Pi)$. The shape Π might not have any corners, but if we take X to be all the points on its surface then the symmetry group of Π is a subgroup of $\text{Sym}(X)$.

Maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ that preserve distance are called *isometries*. Many symmetry groups consist of isometries. For this reason we will often say the *symmetry group of isometries of* Π instead of the *symmetry group of* Π.

Example 3.3.2. Let G be the symmetry group of isometries of a cube $\mathcal{C}$. What does it look like?

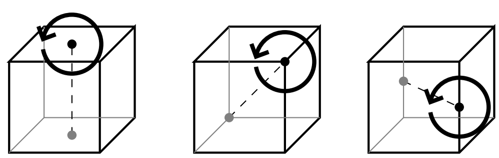


- Notice that any symmetry of a cube is a permutation of its corners
- Choose a face F of the cube. Notice that any permutation of the corners of the cube is completely determined by what happens to the corners of F.
- The symmetry group of F is D_8 (because it is a square)
- There are 6 possible faces that F could be sent to. These can be achieved by 3-dimensional rotations of the cube. Let's call these rotations $R_1, \dots, R_6$ (where R_1 is the trivial rotation)
- By applying a symmetry from D_8 to F, and then applying one of $R_1, \dots, R_6$, we can achieve every symmetry of our cube, and since all these send the corners of F to different positions, they are all different symmetries of the cube
- Hence there are $6 \times |D_8| = 48$ symmetries to the cube, and it's symmetry group is: $\text{Isom}(\mathcal{C}) = \text{D}_8\text{R}_1 \cup \dots \cup \text{D}_8\text{R}_6$. Some of these symmetries are reflections.

There is a natural subgroup of $\text{Isom}(\mathcal{C})$: the group $\text{Rot}(\mathcal{C})$ of rotations of $\mathcal{C}$. Our face F has the 4 rotations e, ρ, ρ^2, ρ^3 from D_8, and there are 6 rotations of the cube that move F to each of the six faces of $\mathcal{C}$. Combining these gives the 24 rotations of our cube. Hence,

$$|\text{Rot}(\mathcal{C})| = 24.$$

It's hard to picture these rotations. Let's try to describe them another way — this will be easier since we know we are looking for exactly 24 rotations. Here are some symmetries in $\text{Rot}(\mathcal{C})$:



(B) & (C) (D) (E)

(A) One identity rotation that does nothing

(Fixes all faces, corners, edges)

(B) Six 90^0 face rotations (One per face. Induces a 270^0 rotation for opposite face)

(Fixes precisely two faces, and no corners, no edges)

(C) Three 180^0 face rotations (One per face but this double-counts so $6/2 = 3$)

(Fixes precisely two faces, and no corners, no edges)

(D) Eight 120^0 corner rotations (One per corner. Induces a 240^0 rotation for opposite corner)

(Fixes precisely two corners)

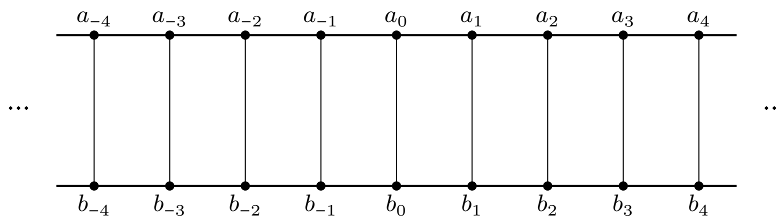
(E) Six 180^0 edge rotations, rotating an edge about it's midpoint

(Fixes precisely two edges)

These are all clearly different (because of the things they fix/don't fix, etc) and there are 24 of them, so they must be all of $\text{Rot}(\mathcal{C})$.

3.3.1 Handout for Section 3.3

Example 3.3.3. We have seen finite isometry groups. Here is an infinite one. Let $\mathcal{S}$ be the following infinite shape, and let G be the symmetry group of isometries of $\mathcal{S}$.



Note that G contains a translation, for example:

$$\rho = (\dots\ a_{-2}\ a_{-1}\ a_0\ a_1\ a_2\ \dots)(\dots\ b_{-2}\ b_{-1}\ b_0\ b_1\ b_2\ \dots)$$

It also contains three types of reflections: a reflection through a horizontal line:

$$\sigma = \cdots(a_{-2}\ b_{-2})(a_{-1}\ b_{-1})(a_0\ b_0)(a_1\ b_1)(a_2\ b_2)\cdots$$

and a vertical reflection through an edge:

$$\theta_1 = (a_{-1}\ a_1)(b_{-1}\ b_1)(a_{-2}\ a_2)(b_{-2}\ b_2)\cdots$$

and a vertical reflection not through an edge:

$$\theta_2 = \cdots(a_{-1}\ a_0)(b_{-1}\ b_0)(a_{-2}\ a_1)(b_{-2}\ b_1)(a_{-3}\ a_2)(b_{-3}\ b_2)\cdots$$

It also has other translations, but these are all obviously powers of ρ. It also has other reflections, but these are all obviously obtained by combining powers of ρ with θ_1 and θ_2.
Hence one can deduce that $G = \langle \rho, \sigma, \theta_1, \theta_2 \rangle$.