13 Solutions to exercises

13.1 Solutions to Exercises 1 - Exercises on permutations

Solution. (Question 1.4.1)

(a) $(3\,2\,7)(4\,1) \in S_7$

(b) $(2\,5)(3\,1) \in S_5$

(c)
$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 11 & 8 & 10 & 4 & 3 & 2 & 6 & 1 & 5 & 9 & 12 \end{matrix}$$

Solution. (Question 1.4.2)

(a) $k^{-1} = (2\ 3\ 1)(9\ 7\ 5)$ (but remember there are lots of equivalent ways of writing this because a cycle $(a\ b\ c)$ can also be written as $(c\ a\ b)$ and $(b\ c\ a)$)

Check: $k(2\ 3\ 1)(9\ 7\ 5) = (1)(2)(3)(4)(5)(6)(7)(8)(9) = e$ so our answer is correct.

(b) $ghk = (1\ 4\ 2\ 7\ 8)(1\ 4\ 5)(5\ 7\ 9)(1\ 3\ 2) = (1\ 3\ 7\ 9\ 4\ 5\ 8)(2)(6) = (1\ 3\ 7\ 9\ 4\ 5\ 8)$.

(c) For $k^{-1}ghk$ we know k^{-1} and we know ghk, so: $k^{-1}ghk = (2\,3\,1)(9\,7\,5)(1\,3\,7\,9\,4\,5\,8) = (1)(2\,3\,5\,8)(4\,9)(6)(7) = (2\ 3\ 5\ 8)(4\ 9)$

Solution. (Question 1.4.3)

(a) 5, because $(1\ 2\ 3\ 4\ 5)^k \neq e$ for $1 \le k \le 4$, but $(1\ 2\ 3\ 4\ 5)^5 = e$

(b) 3

(c) 6

(d) There are many answers. Any product of <u>disjoint</u> cycles, where one is a 5-cycle and the other is a 3-cycle will do. E.g. $(1\ 2\ 3\ 4\ 5)(6\ 7\ 8)$.

(e) No. See (f) for the formula for the order of a permutation. One can see on the formula that there is no way to get 19 in S_{10} because we can only form cycles of length at most 10, and 19 is prime.

(f) Proposition: $o(g) = \mathrm{lcm}(\mathrm{r}_1, \mathrm{r}_2, \ldots, \mathrm{r}_\mathrm{m})$
Proof: Since each c_i is a cycle of length r_i, one can easily check that $o(c_i) = r_i$. Now disjoint cycles commute, so
$$g^k = c_1^k c_2^k \cdots c_m^k.$$
Hence if $\ell = \mathrm{lcm}(\mathrm{r}_1, \mathrm{r}_2, \ldots, \mathrm{r}_\mathrm{m})$, then $g^\ell = ee\cdots e = e$. Thus $o(g) \le \ell$.
On the other hand, for all $1 \le k < \ell$ there is some i for which k is not a multiple of r_i. But then $c_i^k \neq e$. Because the cycles are disjoint, we know that c_i^k is not cancelled out by any other cycle. Therefore $g^k \neq e$. Thus $o(g) \ge \ell$.

Solution. (Question 1.4.4) Since $(1\ 3\ 2)\sigma = (1\ 2)(3\ 4)$ we can **left** multiply by the inverse of $(1\ 3\ 2)$ so obtain:

$$\begin{aligned} (1\ 3\ 2)\sigma &= (1\ 2)(3\ 4) \\ (1\ 3\ 2)^{-1}(1\ 3\ 2)\sigma &= (1\ 3\ 2)^{-1}(1\ 2)(3\ 4) \\ \sigma &= (1\ 3\ 2)^{-1}(1\ 2)(3\ 4). \end{aligned}$$

From lectures we recall how to calculate inverses of permutations using the Quick Inverse Proposition: $(1\ 3\ 2)^{-1} = (2\ 3\ 1)$. Hence:

$$\sigma = (1\ 3\ 2)^{-1}(1\ 2)(3\ 4) = (2\ 3\ 1)(1\ 2)(3\ 4) = (1\ 3\ 4)(2) = (1\ 3\ 4).$$

Now we check we haven't made a mistake by substituting σ into the original equation.

$$(1\ 3\ 2)\sigma = (1\ 3\ 2)(1\ 3\ 4) = (1\ 2)(3\ 4).$$

Success!

Solution. (Question 1.4.5) Here we use the same method as Question 1.4.4.

$$\begin{aligned}(1\ 3\ 2)(5\ 7\ 4)\sigma &= (1\ 2)(3\ 4)\\ \sigma &= \big((1\ 3\ 2)(5\ 7\ 4)\big)^{-1}(1\ 2)(3\ 4)\\ \sigma &= (2\ 3\ 1)(4\ 7\ 5)(1\ 2)(3\ 4)\\ \sigma &= (1\ 3\ 7\ 5\ 4)(2)(6)\\ \sigma &= (1\ 3\ 7\ 5\ 4).\end{aligned}$$

Check:

$$(1\ 3\ 2)(5\ 7\ 4)(1\ 3\ 7\ 5\ 4) = (1\ 2)(3\ 4)(5)(6)(7) = (1\ 2)(3\ 4).$$

Success!

Solution. (Question 1.4.6) Here we again use the same method as Question 1.4.4.

$$\begin{aligned}(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)\sigma &= (9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)\\ \sigma &= (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)^{-1}(9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)\\ \sigma &= (9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)(9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1)\\ \sigma &= (1\ 8\ 6\ 4\ 2\ 9\ 7\ 5\ 3).\end{aligned}$$

Check:

$$(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)(1\ 8\ 6\ 4\ 2\ 9\ 7\ 5\ 3) = (1\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2).$$

This looks like our solution is wrong. But then we remember that there are many different ways of writing the same permutation in cycle notation, and in particular:

$$(1\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2) = (9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1).$$

Hence $\sigma = (1\ 8\ 6\ 4\ 2\ 9\ 7\ 5\ 3)$ really does satisfy our permutation equation.

Success!