13.10 Solutions to Exercises 10 - Exercises on Sylow's Theorems

Solution. (Question 10.4.1)

(j) Let G be a group of order $48600 = 3^5 2^3 5^2$. By Sylow Thm 1, G contains a 5-Sylow subgroup. By definition, any 5-Sylow subgroup of G has order $5^2 = 25$.

(j) $|S_{12}| = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$. By Sylow Thm 1, S_{12} has a 3-Sylow subgroup. By definition, any 3-Sylow subgroup of G has order $3^5 = 243$.

(j) If G has order $726 = 2 \cdot 3 \cdot 11^2$, then by Sylow Thm 1, G has an 11-Sylow subgroup, and by definition this has order $11^2 = 121$.

Solution. (Question 10.4.2) Now $|S_4| = 4! = 2^3 \cdot 3$, so any 2-Sylow subgroup of S_4 has order $2^3 = 8$. Any 3-Sylow subgroup of S_4 has order 3.

The 3-Sylow subgroup is easy to find: take any 3-cycle, and the group it generates will have order 3. E.g. $\langle(134)\rangle$ is a 3-Sylow subgroup of S_4.

To find a 2-Sylow subgroup of S_4 we need to find a subgroup of order 8. Now the dihedral group D_8 has order 8 and (because it is the symmetry group of a square) lies in S_4. Therefore D_8 is a 2-Sylow subgroup of S_4. Any other subgroup of order 8 will also be a correct answer here.

Solution. (Question 10.4.3) Such a group cannot exist. Suppose G is such a group. Then $|G| = 2^3 3$. By Sylow's Thm 3, the number $a(2)$ of 2-Sylow subgroups of G satisfies $a(2) \cong 1 \mod 2$. Hence $a(2) \neq 4$, because $4 \ncong 1 \mod 2$.

Solution. (Question 10.4.4) Now $|S_8| = 8! = 2^7 \cdot 3^2 \cdot 5^1 \cdot 7^1$. The 7-Sylow subgroups of S_8 all have order $7^1 = 7$. Let $a(7)$ denote the number of 7-Sylow subgroups in S_8. We know from Sylow's Thm 3 that $a(7) \cong 1 \mod 7$. Therefore $a(7) \in \{1, 8, 15, ...\}$.
Any subgroup of order 7 is a 7-Sylow subgroup of S_8. The groups,

$$H_1 = \langle(1\ 2\ 3\ 4\ 5\ 6\ 7)\rangle \text{ and}$$
$$H_2 = \langle(2\ 3\ 4\ 5\ 6\ 7\ 8)\rangle$$

are both subgroups of S_8. Since they are cyclic, their order is equal to the order of their generators, so they both have order 7. Hence, H_1 and H_2 are 7-Sylow subgroups of G.
Since G has at least two 7-Sylow subgroups, we know $a(7) \geq 2$. Since $a(7) \in \{1, 8, 15, ...\}$, we must have that $a(7) \geq 8$.

The above is enough to answer the question. If you want to go further with this, you might also note that by Sylow's Third Theorem we also know $a(7)$ divides 5760. Listing all the divisors of 5760 that are congruent to $1 \mod 7$ we have:

$$1, 8, 15, 36, 64, 120, 288, 960.$$

Of course we know 1 is not possible because there is more than one 7-Sylow subgroup. Obviously you will need to know more to reduce this list further.

Solution. (Question 10.4.5) Write $|G| = p^r k$, where k is coprime to p. Let H be a p-Sylow subgroup of G. Then $|H| = p^r$.

For all $g \in G$, it is easy to check that $g^{-1}Hg$ is a subgroup of G. Also, $|g^{-1}Hg| = |H| = p^r$, so $g^{-1}Hg$ is a p-Sylow subgroup of G. However, there is only one p-Sylow subgroup of G. Therefore we must have that $g^{-1}Hg = H$.

Hence, for all $g \in G$ we have $g^{-1}Hg = H$. Therefore $H \trianglelefteq G$.

Solution. (Question 10.4.6) First note that $|G| = 35 = 5^1 7^1$. By Lagrange's Theorem, any subgroup of G can *only* have order dividing 5×7. Hence any subgroup of G has order $1, 5, 7$ or $5 \times 7 = 35$.

By Sylow Thm 1, G has a 5-Sylow subgroup and a 7-Sylow subgroup, and because the highest power of 5 and 7 in $|G|$ is one, this means that G has a subgroup of order 5 and another of order 7, and both of these must be cyclic by Lagrange's theorem.

The number of 5-subgroups is $a(5)$, and by Sylow Thm 3, $a(5) \mid 7$ and $a(7) \mid 5$. Hence $a(5) \in \{1, 7\}$ and $a(7) \in \{1, 5\}$. Furthermore, $a(5) \cong 1 \mod 5$ and $a(7) \cong 1 \mod 7$. Since $7 \ncong 1 \mod 5$ we cannot have $a(5) = 7$, so it must be that $a(5) = 1$. Similarly, $a(7) = 1$.

Hence G has only one subgroup of order 5, and one subgroup of order 7. Thus, the list of subgroups of G is the list of subgroups is $\{\langle 1 \rangle, C_5, C_7, G\}$.

Solution. (Question 10.4.7) Let $a(p)$ denote the number of p-Sylow subgroups in G. First notice that $221 = 13 \times 17$, and both these numbers are prime. By Lagrange, every subgroup of G has order dividing $|G|$. Hence the *possible* subgroup sizes are $1, 13, 17, 221$, since these are the only divisors of 221.

Now $|G| = 13^1 \cdot 17^1$, so any 13-Sylow subgroup of G has order $13^1 = 13$. Similarly, any 17-Sylow subgroup of G has order 17.

By Sylow's Thm 3, $a(13) \mid 17$ and so $a(13)$ equals 1 or 17. Again by Sylow Thm 3, $a(13) \equiv 1 \mod 13$. Now $17 \not\equiv 1 \mod 13$, therefore $a(13) \neq 17$. Hence $a(13) = 1$. This means that G has only one 13-Sylow subgroup. We proved earlier that any 13-Sylow subgroup of G has order 13. Hence G has only one subgroup of order 13. Call this subgroup H.

We now repeat this argument to find $a(17)$. Since $a(17) \mid 13$, we have $a(17)$ equals 1 or 13. Again by Sylow Thm 3, $a(17) \equiv 1 \mod 17$. Since $13 \not\equiv 1 \mod 17$, we have $a(17) = 1$. Hence G has only one subgroup of order 17. Call this subgroup K.

Remember that G can only have subgroups of order 1, 13, 17 and 221. Of course G has only one subgroup of order 1 (it is the trivial subgroup $\langle 1 \rangle$), and one subgroup of order $|G| = 221$ (it is the whole group G). Therefore we have found all the subgroups of G, they are: $\langle 1 \rangle, H, K, G$.

Now $|\langle 1 \rangle \cup H \cup K| \leq 1 + 13 + 17 = 31 < 221$, so there exists some element $g \in G$ that does not lie in $\langle 1 \rangle \cup H \cup K$. In other words, g does not lie in any proper subgroup of G. Hence $\langle g \rangle$ does not equal $\langle 1 \rangle$ or H or K. But $\langle g \rangle$ is a subgroup of G, and we know all the subgroups of G. Hence $\langle g \rangle = G$. In other words, G is cyclic.