10.4 Exercises 10- Exercises on Sylow's Theorems

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Question 10.4.1. Prove the following:

(a) Every group of order $48,600$ contains a subgroup of order 25.

(b) Prove that S_{12} has a subgroup of order 243.

(c) Prove that a group of order 726 has a subgroup of order 121.

Question 10.4.2. Find a 2-Sylow subgroup and a 3-Sylow subgroup of S_4.

Question 10.4.3. Does there exist a finite group of order 24 that has precisely four distinct Sylow 2-subgroups? Give an example or give a proof that no example exists.

Question 10.4.4. Prove that S_8 has at least eight 7-Sylow subgroups.

Question 10.4.5. Let G be a finite group, and let $a(p)$ denote the number of p-Sylow subgroups of G. Prove that if $a(p) = 1$, then the p-Sylow subgroup of G is normal in G.

Question 10.4.6. Let G be a group of order 35. Find all the subgroups of G.

Question 10.4.7. (Hard) Let G be a finite group of order 221. Prove that G must be cyclic.

Hint. (Question 10.4.1) Use Sylow's First Theorem!

Hint. (Question 10.4.2) First work out the orders of these Sylow subgroups. Can you think of any subgroups of S_4 with those orders?

Hint. (Question 10.4.3) Can you think of a theorem that tells you information about the **number** of Sylow subgroups?

Hint. (Question 10.4.4) Can you think of a theorem that helps you calculate the number of Sylow subgroups? Can you think of any 7-Sylow subgroups yourself?

Hint. (Question 10.4.5) Let H be the p-Sylow subgroup of G. For all $g \in G$, prove that $|g^{-1}Hg| = p$. What can you deduce from this?

Hint. (Question 10.4.6) First use Lagrange's Theorem, then use Sylow's Theorems.

Hint. (Question 10.4.7) Work out all the subgroups of G, then see how many elements of G lie in a proper subgroup.