13.11 Solutions to Exercises 11 - Exercises on abelian groups

Solution. (Question 11.4.1) Let $\theta : G \times H \to H \times G$ be the map $\theta((g,h)) = (h,g)$. We must check that θ is one-to-one, onto and a homomorphism.

Choose two elements (g_1,h_1) and (g_2,h_2) in $G \times H$ and recall that $(g_1,h_1)(g_2,h_2) = (g_1g_2, h_1h_2)$.

- (θ is a homomorphism.) First we find $\theta((g_1,h_1)(g_2,h_2)) = \theta((g_1g_2,h_1h_2)) = (h_1h_2, g_1g_2)$. Second we find $\theta((g_1,h_1))\theta((g_2,h_2)) = (h_1,g_1)(h_2,g_2) = (h_1h_2,g_1g_2)$. Therefore, $\theta((g_1,h_1)(g_2,h_2)) = \theta((g_1,h_1))\theta((g_2,h_2))$ and θ is a homomorphism.
- (θ is one-to-one.) If $\theta((g_1,h_1)) = \theta((g_2,h_2))$ then $(h_1,g_1) = (h_2,g_2)$, and thus $h_1 = h_2$ and $g_1 = g_2$. We therefore have $(g_1,h_1) = (g_2,h_2)$.
- (θ is onto.) Any $(h,g) \in H \times G$ satisfies $(h,g) = \theta((g,h))$.

Solution. (Question 11.4.2) Here we use the Quick Subgroup Test.

- Identity: Now $K \leq G$ so $e_G \in K$ and similarly $e_H \in J$. Therefore $(e_G,e_H) \in K \times J$ and (e_G,e_H) is the identity of $G \times H$.
- Inverse: If $(g,h) \in K \times J$, then $g \in K$ and $h \in J$ and therefore (because K and J are groups) $g^{-1} \in K$ and $h^{-1} \in J$. Thus $(g^{-1},h^{-1}) \in K \times J$. Finally, we check that this is the inverse of (g,h). Indeed it is: $(g,h)(g^{-1},h^{-1}) = (gg^{-1}, hh^{-1}) = (e_G,e_H)$.
- Closure: If $(g_1,h_1) \in K \times J$ and $(g_2,h_2) \in K \times J$, then (in $G \times H$) we know

  $$(g_1,h_1)(g_2,h_2) = (g_1g_2,h_1h_2) \in G \times H.$$

  Since K and J are groups, and $g_i \in K$ and $h_i \in J$, we have $g_1g_2 \in K$ and $h_1h_2 \in J$. Therefore,

  $$(g_1,h_1)(g_2,h_2) = (g_1g_2,h_1h_2) \in K \times J.$$

Solution. (Question 11.4.3) The proof roughly follows this plan: we find a cyclic subgroup H of C_{p^r} that has order p^s. Then $H \cong C_{p^s}$. We will find H by finding an element $h \in C_{p^r}$ that has order p^s, and setting $H = \langle h \rangle$. If $s = 0$ or $s = r$ then there is nothing to prove, so assume $1 \leq s < r$.

Let τ be the p^r-cycle $\tau = (1\ 2\ \cdots\ p^r)$. Then $C_{p^r} = \langle \tau \rangle$. Set $h = \tau^{r-s}$. We claim that $o(h) = p^s$. Indeed $h^{p^s} = (\tau^{p^{r-s}})^{p^s} = \tau^{p^{r-s} \cdot p^s} = \tau^{p^r} = e$. Therefore $o(h) \leq p^s$. On the other hand, $o(h) \geq p^s$ since if $0 < \ell < p^s$ then $h^\ell = \tau^{p^{r-s}\ell} \neq e$ because $p^{r-s}\ell < p^r = o(\tau)$. Thus $o(\tau^k) = p^s$.
Finally, we note that $H = \langle h \rangle \leq C_{p^r}$ is a cyclic group of order p^s and is therefore isomorphic to C_{p^s}.

Solution. (Question 11.4.4) By the Fundamental Theorem of Finite Abelian Groups, every abelian group of order 72 is of the form

$$C_{p_1^{e_1}} \times \cdots \times C_{p_n^{e_n}},$$

where $p_1, \ldots, p_n$ are (not necessarily distinct) primes and $e_1, \ldots, e_n$ are natural numbers and $p_1^{e_1} \cdots p_n^{e_n} = 72$. Each factorisation is unique up to reordering of the factors (i.e. reordering factors doesn't matter, but different factorisations result in non-isomorphic groups).

Therefore we can find, up to isomorphism, all abelian groups of order $72 = 2^3 \cdot 3^2$ by thinking of all the ways of writing 72 as $p_1^{e_1} \cdots p_n^{e_n}$ where the p_i are not necessarily distinct primes. As long as we are systematic about it, this is easy. Remember we don't care about the order of the primes.

Shape	Decomposition of $2^3 \cdot 3^2$	Group
$2:(3), \quad 3:(2)$	$2^3 \cdot 3^2$	$C_{2^3} \times C_3$
$2:(3), \quad 3:(1,1)$	$2^3 \cdot 3 \cdot 3$	$C_{2^3} \times C_3 \times C_3$
$2:(2,1), \quad 3:(2)$	$2^2 \cdot 2 \cdot 3^2$	$C_{2^2} \times C_2 \times C_{3^2}$
$2:(2,1), \quad 3:(1,1)$	$2^2 \cdot 2 \cdot 3 \cdot 3$	$C_{2^2} \times C_2 \times C_3 \times C_3$
$2:(1,1,1), \quad 3:(2)$	$2 \cdot 2 \cdot 2 \cdot 3^2$	$C_2 \times C_2 \times C_2 \times C_{3^2}$
$2:(1,1,1), \quad 3:(1,1)$	$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3$	$C_2 \times C_2 \times C_2 \times C_3 \times C_3$

Hence, the abelian groups of order 72 up to isomorphism are:

$$C_{2^3} \times C_3, \quad C_{2^3} \times C_3 \times C_3, \quad C_{2^2} \times C_2 \times C_{3^2}, \quad C_{2^2} \times C_2 \times C_3 \times C_3,$$
$$C_2 \times C_2 \times C_2 \times C_{3^2}, \quad C_2 \times C_2 \times C_2 \times C_3 \times C_3$$

Solution. (Question 11.4.5) This is true, but it only holds for abelian groups! Here is the proof. By The Fundamental Theorem of Abelian Groups, there are (not necessarily distinct) primes $p_1, \ldots, p_n$ and natural numbers $e_1, \ldots, e_n$ such that $p_1^{e_1} \cdots p_n^{e_n} = m$ and

$$G \cong C_{p_1^{e_1}} \times \cdots \times C_{p_n^{e_n}}.$$

Notice that, since $p_1^{e_1} \cdots p_n^{e_n} = m$ and $k \big| m$, the number k must also be a product of (some or possibly all of) these primes, so we can write,

$$k = p_1^{d_1} \cdots p_n^{d_n},$$

for some integers $0 \leq d_i \leq e_i$ for $i = 1, \ldots, n$.
For each $i \in \{1, \ldots, n\}$ we can use Question 11.4.3 to deduce that $C_{p_i^{d_i}} \lesssim C_{p_i^{e_i}}$. Therefore by Question 11.4.2,

$$C_{p_1^{d_1}} \times \cdots \times C_{p_n^{d_n}} \lesssim C_{p_1^{e_1}} \times \cdots \times C_{p_n^{e_n}} \cong G.$$

Since $C_{p_1^{d_1}} \times \cdots \times C_{p_n^{d_n}}$ has order $p_1^{d_1} \cdots p_n^{d_n} = k$, it follows that G has a subgroup of order k.