

13.3 Solutions to Exercises 3 - Exercises on groups arising from geometry

Solution. (Question 3.4.1) To solve $(1\ 2\ 3)(4\ 5\ 6\ 7\ 8) = (1\ 5\ 7\ 8)\sigma(1\ 3\ 4)$ in S_8 , we just have to remember to be careful with our left and right multiplication.

$$\begin{aligned}(1\ 2\ 3)(4\ 5\ 6\ 7\ 8) &= (1\ 5\ 7\ 8)\sigma(1\ 3\ 4) \\(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)(4\ 3\ 1) &= (1\ 5\ 7\ 8)\sigma(1\ 3\ 4)(4\ 3\ 1) \\(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)(4\ 3\ 1) &= (1\ 5\ 7\ 8)\sigma \\(8\ 7\ 5\ 1)(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)(4\ 3\ 1) &= (8\ 7\ 5\ 1)(1\ 5\ 7\ 8)\sigma \\(8\ 7\ 5\ 1)(1\ 2\ 3)(4\ 5\ 6\ 7\ 8)(4\ 3\ 1) &= \sigma \\(2\ 3)(4\ 8)(5\ 6) &= \sigma.\end{aligned}$$

Check your answer: $(1\ 2\ 3)(4\ 5\ 6\ 7\ 8) = (1\ 5\ 7\ 8)(2\ 3)(4\ 8)(5\ 6)(1\ 3\ 4)$.

Solution. (Question 3.4.2) One can see (by drawing a picture) that $\rho\sigma$ is a reflection, which means that its order is 2 and it is its own inverse. Hence

$$(\rho\sigma)^2 = e.$$

But this means $\rho\sigma\rho = e$. Hence (left multiplying by the inverse of ρ followed by the inverse of σ),

$$\rho\sigma = \sigma^{-1}\rho^{-1}.$$

However, σ has order 2, so it is also its own inverse. Hence $\sigma^{-1} = \sigma$. We therefore have $\rho\sigma = \sigma\rho^{-1}$. Finally, we note that $\rho\rho^{n-1} = \rho^n = e$, so $\rho^{-1} = \rho^{n-1}$.

Solution. (Question 3.4.3)

- (a) We might need to draw a picture of a pentagon here, whose corners are labelled 1–5 anticlockwise. Then:

$$\rho = (1\ 2\ 3\ 4\ 5)$$

$$\sigma = (2\ 5)(3\ 4)$$

Other rotations are: e , $(1\ 3\ 5\ 2\ 4)$, $(1\ 4\ 2\ 5\ 3)$, $(1\ 5\ 4\ 3\ 2)$

Other reflections are: $(1\ 3)(4\ 5)$, $(1\ 5)(2\ 4)$, $(1\ 2)(3\ 5)$, $(1\ 4)(2\ 3)$

- (b) By calculating products of permutations, we can see that:

$$\rho^2 = (1\ 3\ 5\ 2\ 4), \quad \rho^3 = (1\ 4\ 2\ 5\ 3), \quad \rho^4 = (1\ 5\ 4\ 3\ 2).$$

$$\sigma\rho = (1\ 5)(2\ 4), \quad \sigma\rho^2 = (1\ 4)(2\ 3), \quad \sigma\rho^3 = (1\ 3)(4\ 5), \quad \sigma\rho^4 = (1\ 2)(3\ 5).$$

$$\text{Hence } D_{10} = \{e, \rho, \rho^2, \rho^3, \rho^4, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3, \sigma\rho^4\}.$$

Solution. (Question 3.4.4) We see that there are (as usual) rotations and reflections. Any symmetry of \mathcal{S} must leave the central square unchanged, and this observation helps us see there are four rotations and four reflections.

The rotations are anticlockwise by 90, 180, 270 and 360 degrees. If ρ is the rotation anticlockwise by 90 degrees, then:

$$\rho = (1\ 3\ 5\ 7)(2\ 4\ 6\ 8).$$

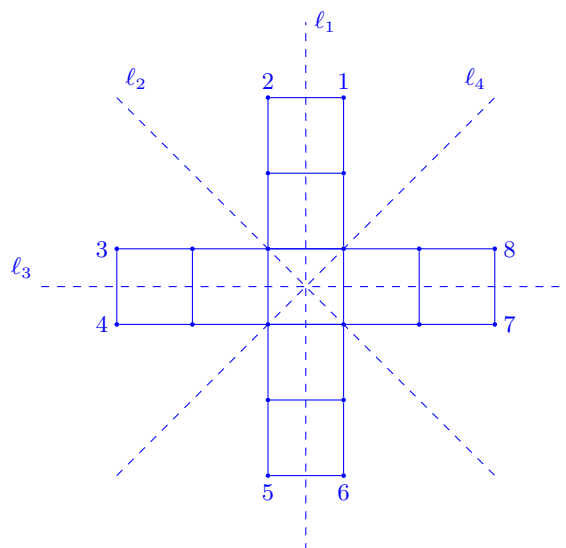
All the rotations are then: e, ρ, ρ^2, ρ^3 , where

$$\rho = (1\ 3\ 5\ 7)(2\ 4\ 6\ 8)$$

$$\rho^2 = (1\ 5)(3\ 7)(2\ 6)(4\ 8)$$

$$\rho^3 = (1\ 7\ 5\ 3)(2\ 8\ 6\ 4)$$

We also have reflections. There are lines of reflectional symmetry pass through the four corners of the central square and through each edge of the central square, as indicated below. Any other reflection would move the central square.



As a permutation, the reflection through ℓ_1 interchanges 1 and 2, interchanges 5 and 6, interchanges 3 and 8, and interchanges 4 and 7. Hence we can write it as,

$$\sigma_1 = (1\ 2)(3\ 8)(4\ 7)(5\ 6).$$

Similarly, we have

$$\sigma_2 = (1\ 4)(2\ 3)(6\ 7)(5\ 8)$$

$$\sigma_3 = (1\ 6)(2\ 5)(3\ 4)(7\ 8)$$

$$\sigma_4 = (1\ 8)(2\ 7)(3\ 6)(4\ 5).$$

Hence the symmetry group of \mathcal{S} is $\{e, \rho, \rho^2, \rho^3, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, which is equal to:

$$\{e, (1\ 3\ 5\ 7)(2\ 4\ 6\ 8), (1\ 5)(3\ 7)(2\ 6)(4\ 8), (1\ 7\ 5\ 3)(2\ 8\ 6\ 4), \\ (1\ 2)(3\ 8)(4\ 7)(5\ 6), (1\ 4)(2\ 3)(6\ 7)(5\ 8), (1\ 6)(2\ 5)(3\ 4)(7\ 8), (1\ 8)(2\ 7)(3\ 6)(4\ 5)\}.$$

Solution. (Question 3.4.5) There are infinitely many possible answers here, so there is no model solution. An natural example is to draw a hexagon, then on each edge draw a square in the obvious way (so it looks like six squares glued onto a hexagon).

Solution. (Question 3.4.6) Recall that an element in $\langle \rho, \sigma \rangle$ looks like:

$$g = \rho^{n_1} \sigma^{k_1} \rho^{n_2} \sigma^{k_2} \dots \rho^{n_m} \sigma^{k_m}$$

for some $n_i, k_i \in \mathbb{Z}$ and some $m \in \mathbb{N}$.

Using property (ii), we can “move” the ρ and σ around, to give us:

$$\begin{aligned} g &= \rho^{n_1} \sigma^{k_1} \rho^{n_2} \sigma^{k_2} \dots \rho^{n_m} \sigma^{k_m} \\ &= \sigma^{k_1} \rho^{-n_1} \rho^{n_2} \sigma^{k_2} \dots \rho^{n_m} \sigma^{k_m} \\ &= \sigma^{k_1} \rho^{n_2 - n_1} \sigma^{k_2} \dots \rho^{n_m} \sigma^{k_m} \\ &= \sigma^{k_1} \sigma^{k_2} \rho^{-(n_2 - n_1)} \dots \rho^{n_m} \sigma^{k_m} \\ &= \dots \\ &= \sigma^k \rho^n \quad \text{for some } k, n \in \mathbb{Z} \end{aligned}$$

However, $\sigma^2 = e$ and $\rho^5 = e$, so we only need to take $k = 0, 1$ and $n = 0, 1, 2, 3, 4$. Hence:

$$\begin{aligned}\langle \rho, \sigma \rangle &= \{\sigma^0 \rho^0, \sigma^0 \rho^1, \sigma^0 \rho^2, \sigma^0 \rho^3, \sigma^0 \rho^4, \sigma^1 \rho^0, \sigma^1 \rho^1, \sigma^1 \rho^2, \sigma^1 \rho^3, \sigma^1 \rho^4\} \\ &= \{e, \rho, \rho^2, \rho^3, \rho^4, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3, \sigma\rho^4\} \\ &= D_{10}.\end{aligned}$$