3.4 Exercises 3- Exercises on groups arising from geometry

(S)

Question 3.4.1. (Quick recall question) Find a permutation $\sigma \in S_9$ that satisfies the following equation: $(1\ 2\ 3)(4\ 5\ 6\ 7\ 8) = (1\ 5\ 7\ 8)\sigma(1\ 3\ 4)$.

Question 3.4.2. Consider the dihedral group D_{2n}. In the usual notation for dihedral groups, ρ is an anticlockwise rotation by $\frac{2\pi}{n}$ and σ is a reflection.

Prove that $\rho\sigma = \sigma\rho^{-1} = \sigma\rho^{n-1}$.

[Hint: What is the order of $\rho\sigma$? What is the inverse of $\rho\sigma$?]

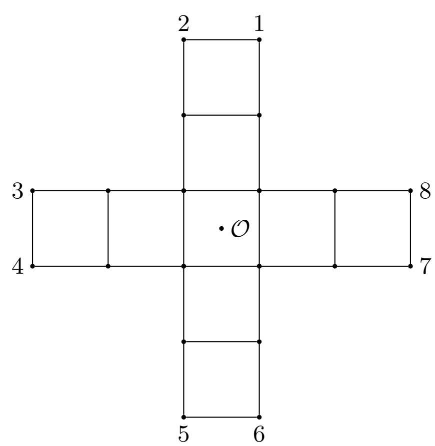
Question 3.4.3. Consider the dihedral group D_{10}. It is the collection of symmetries of a pentagon.
In the usual notation for dihedral groups, label the corners 1–5 anticlockwise and take ρ to be an anticlockwise rotation by $\frac{2\pi}{5}$, and σ to be a reflection through the line of symmetry that passes through the corner labelled 1.

(a) Write down, using cycle notation, the elements in D_{10}. [Hint: draw a diagram]

(b) Verify that the elements of D_{10} can be written as $\{e, \rho, \rho^2, \rho^3, \rho^4, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3, \sigma\rho^4\}$, where all the rotations are of the form ρ^i and all the reflections are of the form $\sigma\rho^i$.

Remark: In the near future, we will use Lagrange's Theorem to quickly prove that every dihedral group D_{2n} can be written as $\{e, \rho, \rho^2, \ldots, \rho^{n-1}, \sigma, \sigma\rho, \sigma\rho^2, \ldots, \sigma\rho^{n-1}\}$, where the rotations are of the form ρ^i and the reflections are of the form $\sigma\rho^i$.

Question 3.4.4. Consider the following shape $\mathcal{S}$ (drawn in the Euclidean plane with centre at the origin $\mathcal{O}$), whose corners have been labelled $1, 2, \ldots, 8$ as shown.



The symmetry group of isometries of $\mathcal{S}$ can be thought of as a subgroup of S_8. Write down (in cycle notation) all elements in the symmetry group of $\mathcal{S}$.

Question 3.4.5. Draw an example of a shape $\mathcal{S}$ *that is not a hexagon*, whose symmetry group of isometries is isomorphic to D_{12}.

Question 3.4.6. Let ρ and σ be (respectively) a nontrivial rotation and reflection in D_{10}. In lectures, we will soon see that $\langle\rho, \sigma\rangle$ means the smallest subgroup of D_{10} containing both ρ and σ.

Using the following two facts, show that $\langle\rho, \sigma\rangle = D_{10}$.

(i) $D_{10} = \{e, \rho, \rho^2, \rho^3, \rho^4, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3, \sigma\rho^4\}$

(ii) $\rho\sigma = \sigma\rho^{-1}$