

## 13.4 Solutions to Exercises 4 - Exercises on normal subgroups, simplicity and Lagrange's Theorem

**Solution.** (Question 4.4.1) We must show that for all  $h \in H$  and all  $k \in H \cap N$  we have that  $h^{-1}kh \in H \cap N$ .

Fix  $k \in H \cap N$  and fix  $h \in H$ . Now  $h^{-1}kh \in H$  because  $h, k \in H$ . Also  $h^{-1}kh \in N$  because  $N$  is normal in  $G$ . Therefore  $h^{-1}kh \in H \cap N$ .

**Solution.** (Question 4.4.2) Well  $|S_5| = 5! = 5 \cdot 3 \cdot 2^3$ . By Lagrange, the order of any subgroup of  $S_5$  must divide the order of  $S_5$ . Since 9 does not divide  $5!$ , there is no subgroup of  $S_5$  of order 9.

**Solution.** (Question 4.4.3) We first show that  $HN \subseteq NH$ . Fix  $hk \in HN$ , with  $h \in H$  and  $k \in N$ . Because  $N$  is normal, we can use the Order Switching Lemma, to deduce that there exists  $k' \in N$  such that  $hk = k'h$ . But  $k'h \in NH$ . Hence  $HN \subseteq NH$ .

To show  $NH \subseteq HN$ , we now reverse the above argument: Fix  $kh \in NH$ . Because  $N$  is normal, there exists  $k'' \in N$  such that  $kh = hk''$ . But  $hk'' \in HN$ . Hence  $NH \subseteq HN$ .

**Solution.** (Question 4.4.4)

- (a) Suppose  $H \leq G$ . By Lagrange's Theorem,  $|H|$  divides  $|G| = p$ . But  $p$  is prime, so  $|H| = 1$  or  $p$ . In other words,  $|H| = 1$  or  $|H| = |G|$ . We know from a previous problem sheet that this implies that  $H = \langle e_G \rangle$  or  $H = G$ .
- (b) To show  $G$  is cyclic, we need to find some  $g \in G$  such that  $G = \langle g \rangle$ .

Choose any nontrivial  $g \in G$ . We know  $\langle g \rangle$  is a subgroup of  $G$ . Because  $g \neq e_G$ , we also know that  $\langle g \rangle \neq \langle e_G \rangle$ . By part (a), the only subgroups of  $G$  are  $\langle e_G \rangle$  and  $G$ . Hence  $\langle g \rangle = G$ .

**Solution.** (Question 4.4.5) We just need to check the conditions of the Quick Subgroup Test: identity, closure, inverse.

- [Identity] Now  $e_G$  lies in  $H$  and  $N$ , and so  $e_G e_G \in HN$ . Hence,  $e_G \in HN$ .
- [Closure] Fix two elements in  $HN$ , by fixing  $h_1, h_2 \in H$  and  $k_1, k_2 \in N$  and considering  $h_1 k_1 \in HN$  and  $h_2 k_2 \in HN$ . Now we want to show  $(h_1 k_1)(h_2 k_2)$  lies in  $HN$ . For this we use the Order Switching Lemma to change the orders.

Now

$$(h_1 k_1)(h_2 k_2) = h_1(k_1 h_2)k_2.$$

We can apply the Order Switching Lemma to  $k_1 h_2$  to deduce there exists  $k'_1 \in N$  such that  $k_1 h_2 = h_2 k'_1$ . Hence,

$$(h_1 k_1)(h_2 k_2) = h_1(k_1 h_2)k_2 = (h_1 h_2)(k'_1 k_2) \in HN.$$

- [Inverse] Fix an element in  $HN$ , by fixing  $h \in H$  and  $k \in N$  and considering  $hk \in HN$ . Now  $(hk)^{-1} = k^{-1}h^{-1}$ . We must show that  $k^{-1}h^{-1} \in HN$ . We can apply the Order Switching Lemma to deduce that there exists  $k' \in N$  such that  $k^{-1}h^{-1} = h^{-1}k'$ . But  $h^{-1}k' \in HN$ , therefore  $(hk)^{-1} \in HN$ .

**Solution.** (Question 4.4.6) Recall that  $S_3 = \{e, (123), (132), (12), (13), (23)\}$  and  $C_3 = \{e, (123), (132)\}$ . In  $S_3$ , the product of a 2-cycle and a 3-cycle is always a 2-cycle (this is easily checked). Moreover, the product of two 2-cycles is either trivial (if the 2-cycles are equal) or it is a 3-cycle (again this is easily checked).

Now we know this, we can check for normality. Fix  $g \in S_3$  and  $c \in C_3$ . We must show that  $g^{-1}cg \in C_3$ .

Note that either  $g \in C_3$ , or  $g$  is a 2-cycle.

- If  $g \in C_3$ , then  $g^{-1}cg \in C_3$  (by closure property of  $C_3$ .)
- If  $g$  is a 2-cycle, then  $g^{-1} = g$  and so  $g^{-1}$  is also a 2-cycle. Hence  $g^{-1}c$  is a 2-cycle. Therefore  $g^{-1}cg = (g^{-1}c)g$  is a product of two 2-cycles and is therefore either trivial or a 3-cycle. In both cases, it lies in  $C_3$ .

Hence,  $g^{-1}cg \in C_3$ , and so  $C_3 \trianglelefteq S_3$ .

**Solution.** (Question 4.4.7)

- This permutation fixes every element in  $\mathbb{Z}$  except 1, 3, 5, 6, 8, 10. Therefore it fixes all but finitely many elements of  $\mathbb{Z}$ , so it lies in  $\text{FS}(\mathbb{Z})$ .
- This permutation fixes only 0, and every other element in  $\mathbb{Z}$  is moved. Hence it does not lie in  $\text{FS}(\mathbb{Z})$ .
- Every element in  $\text{FS}(\mathbb{Z})$  is a permutation of  $\mathbb{Z}$ , and therefore lies in  $\text{Sym}(\mathbb{Z})$ . Hence  $\text{FS}(\mathbb{Z}) \subseteq \text{Sym}(\mathbb{Z})$  and we can use the Quick Subgroup Test:

[Identity]  $e$  fixes all but finitely many elements of  $\mathbb{Z}$  (in fact it fixes all elements) and so  $e \in \text{FS}(\mathbb{Z})$

[Closure] If  $\rho, \tau \in \text{FS}(\mathbb{Z})$ , then they both move only finitely many things in  $\mathbb{Z}$ . Hence their product moves only finitely many things.

[Inverse] If  $\rho \in \text{FS}(\mathbb{Z})$  then  $\rho^{-1}$  is a permutation moving only finitely many things (since it must “undo” those things moved by  $\rho$ ) and so it lies in  $\text{FS}(\mathbb{Z})$ .

- The group is not abelian, since any permutations in  $S_n$  can be thought of as lying in  $\text{FS}(\mathbb{Z})$  and we know there are elements in  $S_n$  that do not commute.
- Suppose  $g \in \text{Sym}(\mathbb{Z})$  and  $h \in \text{FS}(\mathbb{Z})$ . We must show that  $g^{-1}hg \in \text{FS}(\mathbb{Z})$ .

Now let  $X \subseteq \mathbb{Z}$  be the set of elements of  $\mathbb{Z}$  that are not fixed by  $h$ . By definition of  $\text{FS}(\mathbb{Z})$ , we know that  $X$  is finite. This set  $X$  is called the *support* of  $h$ . Now let  $g^{-1}X = \{g^{-1}n : n \in X\}$  and notice that  $X$  and  $g^{-1}X$  contain the same number of elements.

Suppose  $m \notin g^{-1}X$ . Then  $gm \notin X$ . We therefore have that  $hgm = gm$  (this is because  $X$  contains all the things that are moved by  $h$ ). Therefore  $g^{-1}hgm = g^{-1}gm = m$ . In other words,  $g^{-1}hg$  fixes everything that does not lie in  $g^{-1}X$ . Since the set  $g^{-1}X$  is finite, we therefore have that  $g^{-1}hg$  only moves finitely many elements in  $\mathbb{Z}$ , and therefore  $g^{-1}hg \in \text{FS}(\mathbb{Z})$ .