

4.4 Exercises 4- Exercises on normal subgroups, simplicity and Lagrange's Theorem

(S)

Question 4.4.1. Let G be a group with $H \leq G$ and $N \trianglelefteq G$. Prove that $H \cap N$ is a normal subgroup of H .

Question 4.4.2. Prove that S_5 has no subgroup of order 9.

Question 4.4.3. Let G be a group with $H \leq G$ and $N \trianglelefteq G$. Prove that $HN = \{hk : h \in H, k \in N\}$ is equal to $NH = \{kh : k \in N, h \in H\}$.

Question 4.4.4. Let p be a prime and let G be a group with $|G| = p$.

- (a) Prove that the only subgroups of G are $\langle e_G \rangle$ and G
- (b) Hence prove that G is cyclic.

Question 4.4.5. Let G be a group with $H \leq G$ and $N \trianglelefteq G$. Prove that $HN \leq G$.

Question 4.4.6. Show that $C_3 \trianglelefteq S_3$.

Question 4.4.7. Recall that $\text{FS}(\mathbb{Z})$ denotes the finitary permutations of \mathbb{Z} , and consists of permutations of \mathbb{Z} that move only finitely many elements of \mathbb{Z} .

- (a) Does the permutation $(1\ 3\ 5)(6\ 8\ 10)$ lie in $\text{FS}(\mathbb{Z})$?
- (b) Let τ be the permutation which maps each $n \in \mathbb{Z}$ to $-n$. Does τ lie in $\text{FS}(\mathbb{Z})$?
- (c) Prove that $\text{FS}(\mathbb{Z}) \leq \text{Sym}(\mathbb{Z})$.
- (d) Is $\text{FS}(\mathbb{Z})$ abelian?
- (e) Prove that $\text{FS}(\mathbb{Z}) \trianglelefteq \text{Sym}(\mathbb{Z})$.

Note: for a general set X , the group $\text{FS}(X)$ is called the *finitary symmetric group on X* , and is always a normal subgroup of $\text{Sym}(X)$.

Hint. (Question 4.4.1) Show that for all $h \in H$ and all $k \in H \cap N$ we have that $h^{-1}kh \in H \cap N$.

Hint. (Question 4.4.2) Use Lagrange's Theorem.

Hint. (Question 4.4.3) Use the Order Switching Lemma to show $HN \subseteq NH$ and vice versa. Do not assume the group is abelian!

Hint. (Question 4.4.4) To prove (a) use Lagrange's Theorem. To prove (b) use (a) to consider what any cyclic subgroup $\langle g \rangle \leq G$ must look like.

Hint. (Question 4.4.5) Use the Order Switching Lemma to check the conditions of the Quick Subgroup Test. Do not assume the group is abelian!

Hint. (Question 4.4.6) Write out all the elements of S_3 and C_3 , then ask yourself: What is the product of a 2-cycle with a 3-cycle? What is the product of two 2-cycles? What is the product of two 3-cycles? Use this to show normality.