13.5 Solutions to Exercises 5 - Exercises on homomorphisms, isomorphisms and quotient groups

Solution. (Question 5.4.1)

(a) Everything in C_7 is a power of the cycle $\rho = (1\ 2\ 3\ 4\ 5\ 6\ 7)$. So for all $\rho^n, \rho^m \in C_7$ we have $\varphi(\rho^n\rho^m) = (\rho^n\rho^m)^2 = \rho^n\rho^m\rho^n\rho^m = \rho^{2n}\rho^{2m} = \varphi(\rho^n)\varphi(\rho^m)$. So, this is a homomorphism. Be careful — we really used the fact that C_7 is abelian here.

Is it an isomorphism? It's certainly onto since you can easily check that:

$$\text{Im}(\varphi) = \{\rho^{2\times1}, \rho^{2\times2}, \dots, \rho^{2\times7}\} = \text{C}_7.$$

One-to-one is slightly tricker: $\varphi(\rho^n) = \varphi(\rho^m) \implies \rho^{2n} = \rho^{2m} \implies \rho^{2(n-m)} = e \implies 2(n-m)|7 \implies (n-m)|7 \implies \rho^{n-m} = e \implies \rho^n = \rho^m$.

(b) We might think this is a homomorphism at first glance, but when you test it carefully things start to go wrong:

$$\varphi(gh) = (gh)^2 = ghgh \quad \text{but} \quad \varphi(g)\varphi(h) = g^2h^2.$$

Because S_7 is not abelian, it's not true in general that $ghgh = g^2h^2$. All we need to do is find some elements confirming this!

Let $g = (1\,2\,3)$ and $h = (1\,2)$. Then $g^2h^2 = (1\,3\,2)$, while on the other hand $ghgh = (1\,2\,3)(1\,2)(1\,2\,3)(1\,2) = (1)(2)(3) = e$. So, for this pair,

$$\varphi(gh) \neq \varphi(g)\varphi(h).$$

Hence φ is not a homomorphism. Thus certainly not an isomorphism.

Solution. (Question 5.4.2) For any $g \in G$ we have $\theta(g) = \theta(e_G g) = \theta(e_G)\theta(g)$. Right multiplying both sides by $(\theta(g))^{-1}$ we have,

$$\theta(g)\,(\theta(g))^{-1} = \theta(e_G)\theta(g)\,(\theta(g))^{-1}.$$

Hence

$$e_H = \theta(e_G)e_H.$$

Therefore $e_H = \theta(e_G)$.

Solution. (Question 5.4.3) We will handle the $m \in \mathbb{N}$ case first. Fix $g \in G$ and $m \in \mathbb{N}$. Then

$$\begin{aligned}\theta(g^m) &= \theta(g^{m-1})\theta(g)\\ &= \theta(g^{m-2})\theta(g)\theta(g)\\ &= \cdots\\ &= \theta(g)\cdots\theta(g)\theta(g) \qquad (m \text{ times})\\ &= \theta(g)^m.\end{aligned}$$

[You could also easily prove this by induction.]

Solution. (Question 5.4.4) Let $K = \text{Ker}(\theta)$. Fix $a, b \in K$ and note that $\theta(a) = \theta(b) = e_G$ by definition.
Now fix $g \in G$. For all $a \in K$ we have $\theta(g^{-1}ag) = (\theta(g))^{-1}\theta(a)\theta(g) = (\theta(g))^{-1}e_G\theta(g) = e_G$. Hence $g^{-1}ag \in K$ for all $a \in K$. Hence, $K \trianglelefteq G$.

[Optional: To prove K is a subgroup, use the Quick Subgroup Test. Firstly, we have already shown $\theta(e_G) = e_H$, therefore $e_G \in K$. Secondly, $\theta(ab) = \theta(a)\theta(b) = e_Ge_G = e_G$, so $ab \in K$. Thirdly, $\theta(a^{-1}) = (\theta(a))^{-1} = e_G^{-1} = e_G$, so $a^{-1} \in K$. Hence, by the Quick Subgroup Test, $K \leq G$.]

Solution. (Question 5.4.5) Write out all the elements of D_6 and all the elements of S_3:

$$D_6 = \{e, \rho, \rho^2, \sigma, \sigma\rho, \sigma\rho^2\} \quad \text{with } \rho\sigma = \sigma\rho^{-1}, \text{ and } \sigma^2 = e, \text{ and } \rho^3 = e.$$
$$S_3 = \{e, (1\ 2\ 3), (1\ 3\ 2), (1\ 2), (1\ 3), (2\ 3)\}.$$

Now let $\theta : D_6 \to S_3$ be the map given by $\theta(\rho) = (1\ 2\ 3)$ and $\theta(\sigma) = (1\ 2)$.

Then let:

$$\theta(e) = e$$
$$\theta(\rho^2) = \theta(\rho)\theta(\rho) = (1\ 2\ 3)^2 = (1\ 3\ 2)$$
$$\theta(\sigma\rho) = \theta(\sigma)\theta(\rho) = (1\ 2)(1\ 2\ 3) = (2\ 3)$$
$$\theta(\sigma\rho^2) = \theta(\sigma)\theta(\rho^2) = (1\ 2)(1\ 3\ 2) = (1\ 3)$$

We can now easily check that θ is well-defined and is a homomorphism. You can also easily see that it is one-to-one (all images are different) and onto (everything is S_3 lies in the image). Hence this is an isomorphism.