5.4 Exercises 5- Exercises on homomorphisms, isomorphisms and quotient groups

(S)

Question 5.4.1. Are the following maps homomorphisms? Are they isomorphisms? Give some reasons (don't worry about full proofs).

(a) $\varphi : C_7 \to C_7$ given by $\varphi(g) = g^2$ for all $g \in C_7$.

(b) $\varphi : S_7 \to S_7$ given by $\varphi(g) = g^2$ for all $g \in S_7$.

Question 5.4.2. Let G and H be groups, with $\theta : G \to H$ a homomorphism. Prove that $\theta(e_G) = e_H$.

Question 5.4.3. Let G and H be groups, with $\theta : G \to H$ a homomorphism. Prove that $\theta(g^m) = (\theta(g))^m$ holds for all $g \in G$ and all $m \in \mathbb{N}$.

[In fact this result is true for all $m \in \mathbb{Z}$.]

Question 5.4.4. Let G and H be groups, with $\theta : G \to H$ a homomorphism. Prove that $\mathrm{Ker}(\theta) \trianglelefteq \mathrm{G}$.

You may assume without proof that $\mathrm{Ker}(\theta)$ is a subgroup of G.

[Optional: Prove that $\mathrm{Ker}(\theta)$ is a subgroup of G too.]

Question 5.4.5. Prove that $D_6 \cong S_3$.

Hint. (Question 5.4.1) Just play around with these definitions. Choose your favourite few permutations in C_7 and see where φ sends them.

Hint. (Question 5.4.2) Notice that, for any $g \in G$, we have $\theta(g) = \theta(e_G g) = \theta(e_G)\theta(g)$. Can you rearrange this to find $\theta(e_G)$?

Hint. (Question 5.4.3) Notice that $\theta(g^m) = \theta(g^{m-1})\theta(g)$. Now continue!

Hint. (Question 5.4.4) Note that $\theta(g^{-1}ag) = (\theta(g))^{-1}\theta(a)\theta(g)$. Do you know what $\theta(a)$ equals? Yes! Because a lies in the kernel.

Hint. (Question 5.4.5) Write out all the elements of D_6 and all the elements of S_3:

$$D_6 = \{e, \rho, \rho^2, \sigma, \sigma\rho, \sigma\rho^2\} \quad \text{with } \rho\sigma = \sigma\rho^{-1}, \text{ and } \sigma^2 = e, \text{ and } \rho^3 = e.$$
$$S_3 = \{e, (1\ 2\ 3), (1\ 3\ 2), (1\ 2), (1\ 3), (2\ 3)\}.$$

Can you think of an isomorphism θ from D_6 to S_3? Perhaps choose $\theta(\rho)$ and $\theta(\sigma)$, then use this to determine all the other images.