

13.6 Solutions to Exercises 6 - Exercises on the isomorphism theorems

Solution. (Question 6.1.1) Proof: We simply check the conditions in the Quick Subgroup Test. Fix $h_1, h_2 \in \text{Im}(\theta)$. Then there exist g_1, g_2 such that $h_i = \theta(g_i)$ for $i = 1, 2$. Then:

- Since $\theta(e_G) = e_H$, we know $e_H \in \text{Im}(\theta)$ [Identity]
- $\theta(g_1 g_2) = \theta(g_1)\theta(g_2) = h_1 h_2$, which implies $h_1 h_2 \in \text{Im}(\theta)$ [Closure]
- $\theta(g_1^{-1}) = (\theta(g_1))^{-1} = h_1^{-1}$, and therefore $h_1^{-1} \in \text{Im}(\theta)$. [Inverses]

Solution. (Question 6.1.2) As noted the image of the signature function is C_2 . Now let K be the kernel of the signature function. Then, by the First Isomorphism Theorem, we have that $S_n/K \cong C_2$.

Solution. (Question 6.1.3) Fix $h_1, h_2 \in H$. Then $\phi(h_1 h_2) = h_1 h_2 N = (h_1 N)(h_2 N) = \phi(h_1)\phi(h_2)$. Thus ϕ is indeed a homomorphism.

Solution. (Question 6.1.4) To simplify our notation, let $G := GL_2(\mathbb{R})$ and $K := SL_2(\mathbb{R})$.

- (a) We want to show $P \trianglelefteq G$ and $K \trianglelefteq G$. Note that if $M \in G$ and $A \in G$ then $\det(M^{-1}AM) = \det(M)^{-1} \det(A) \det(M)$. Since all these determinants are real numbers, and multiplication order doesn't matter for real numbers, we have that $\det(M^{-1}AM) = \det(A)$. Hence if $A \in P$ then $\det(M^{-1}AM) = \det(A) > 0$ and therefore $M^{-1}AM \in P$; thus $P \trianglelefteq G$. Similarly, if $A \in K$ then $\det(M^{-1}AM) = \det(A) = 1$ and therefore $M^{-1}AM \in K$; thus $K \trianglelefteq G$.
- (b) From linear algebra, we know that the determinant function is a map $\det : G \rightarrow R^*$ where R^* denotes the group of non-zero real numbers under multiplication. It is easy to check this is a homomorphism and its kernel is all matrices in G with determinant 1 — i.e. the kernel is the group K . Hence, by the First Isomorphism Theorem, $G/K \cong R^*$. So, intuitively G/K is the group of non-zero real numbers under multiplication. Similarly, P/K is the group of positive non-zero real numbers under multiplication. You might be able to see intuitively that G/P is therefore basically $\{1, -1\}$ under multiplication (which we know is the same as the cyclic group C_2 by Lagrange's Theorem). If you can't see this immediately then read on. To picture G/P write $J := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in G$ and note that $\det(J) = -1$. Hence $\det(JM) = -\det(M)$. Therefore, for all $M \in G$ either $M \in P$ (if $\det(M) > 0$) or $JM \in P$ (if $\det(M) < 0$). Hence $M = P \cup JP$ and $M/P = \{I, J\}$ where I is the identity matrix. This is a group, because $J^2 = I$, and clearly $\{I, J\} \cong C_2$. Thus, intuitively, $G/P \cong C_2$.
- (c) This follows immediately from the Third Isomorphism Theorem.