6.1 Exercises 6- Exercises on the isomorphism theorems

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Question 6.1.1. Let G and H be groups, with $\theta : G \to H$ a homomorphism. Prove that $\mathrm{Im}(\theta) \leq \mathrm{H}$.

Question 6.1.2. Suppose $n \geq 2$. There exists a surjective homomorphism $\sigma : S_n \to C_2$. Prove that S_n has a normal subgroup K such that $S_n/K \cong C_2$.

[The function σ is called the signature function, and we will study it in detail.]

Question 6.1.3. Suppose G is a group with $H \leq G$ and $N \trianglelefteq G$. Let $\phi : H \to (HN)/N$ be the map given by $\phi(h) = hN$. Prove that ϕ is a homomorphism (we used this fact in the proof of the Second Isomorphism Theorem).

Question 6.1.4. Consider the group $GL_2(\mathbb{R})$ (the group of 2×2 invertible matrices), the group $SL_2(\mathbb{R})$ (the group of 2×2 invertible matrices with determinant equal to 1) and the group $P = \{M \in GL_2(\mathbb{R}) : \det(M) > 0\}$.

(a) Prove that P and $SL_2(\mathbb{R})$ are normal in $GL_2(\mathbb{R})$.

(b) Give an intuitive description of the quotient groups $GL_2(\mathbb{R})/SL_2(\mathbb{R})$ and $P/SL_2(\mathbb{R})$ and $GL_2(\mathbb{R})/P$.

(c) Prove that,

$$\frac{GL_2(\mathbb{R})/SL_2(\mathbb{R})}{P/SL_2(\mathbb{R})} \cong GL_2(\mathbb{R})/P.$$

Hint. (Question 6.1.1) Use the Quick Subgroup Test!

Hint. (Question 6.1.2) Use the signature function as the homomorphism in the First Isomorphism Theorem.

Hint. (Question 6.1.3) Recall the definition of multiplcation in quotient groups: $(h_1N)(h_2N) = h_1h_2N$. Now check the definition of a homomorphism.

Hint. (Question 6.1.4) Recall from linear algebra that if A, B, C are $n \times n$ matrices, then $\det(ABC) = \det(A)\det(B)\det(C)$ and from this of course we have that $\det(A^{-1}) = \det(A)^{-1}$ whenever A is invertible.