13.7 Solutions to Exercises 7 - Exercises on the alternating group

Solution. (Question 7.4.1) Recall from lectures that there is an easy way to calculate the signature of a permutation $g \in S_n$:

- First write g as a product of disjoint cycles: $g = c_1 c_2 \cdots c_m$
- If r_i is the length of the cycle c_i, then $\sigma(c_i) = (-1)^{r_i - 1}$.
- σ is a homomorphism, so $\sigma(g) = \sigma(c_1)\sigma(c_2)\cdots\sigma(c_m) = (-1)^{r_1-1}(-1)^{r_2-1}\cdots(-1)^{r_m-1}$.
- A_n is all even permutations in S_n — that is all permutations in S_n whose signature is equal to 1.

Using these properties, we can easily answer the question.

(a) g is written already as a product of disjoint cycles, so $\sigma(g) = (-1)^{3-1}(-1)^{4-1}(-1)^{3-1} = -1$. Hence $g \notin A_{10}$.

(b) $\sigma(h) = (-1)^{2-1}(-1)^{2-1}(-1)^{2-1}(-1)^{2-1} = 1$. Hence $h \in A_{10}$.

(c) We can calculate g^{-1} and then work out its signature. Or we can be clever and recall that g^{-1} is just g with all its cycles written backwards. Hence when g^{-1} is written as a product of disjoint cycles, all the cycles used have the same lengths as those in g. Hence $\sigma(g^{-1}) = \sigma(g)$.

We have $\sigma(g^{-1}) = -1$, so $g^{-1} \notin A_{10}$.

(d) Since σ is a homomorphism, $\sigma(gh) = \sigma(g)\sigma(h) = (-1)(1) = -1$. Hence $gh \notin A_{10}$.

(e) Again, since σ is a homomorphism, $\sigma(g^{-1}hg) = \sigma(g^{-1})\sigma(h)\sigma(g) = (-1)(1)(-1) = 1$. Hence $g^{-1}hg \in A_{10}$.

Solution. (Question 7.4.2)

(a) A permutation with cycle shape $(3,2,2,2)$ has signature $(-1)^{3-1}(-1)^{2-1}(-1)^{2-1}(-1)^{2-1} = -1$. Hence $(3,2,2,2)$ is not the cycle shape of any permutation in an alternating group.

(b) A permutation with cycle shape $(7,3,3,2,2)$ has signature $(-1)^{7-1}(-1)^{3-1}(-1)^{3-1}(-1)^{2-1}(-1)^{2-1} = 1$. Hence any permutation with cycle shape $(7,3,3,2,2)$ lies in an alternating group.

(c) A permutation with cycle shape $(5,5,5)$ has signature $(-1)^{5-1}(-1)^{5-1}(-1)^{5-1} = 1$. Hence any permutation with cycle shape $(5,5,5)$ lies in an alternating group.

(d) A permutation with cycle shape $(2,2,2,2,2)$ has signature $(-1)^{2-1}(-1)^{2-1}(-1)^{2-1}(-1)^{2-1}(-1)^{2-1} = -1$. Hence any permutation with cycle shape $(2,2,2,2,2)$ does not lie in an alternating group.

Solution. (Question 7.4.3) Consider the following table of cycle shapes in S_3.

Cycle shape	Example of this cycle shape	Signature	In A_3?
$\varnothing$	e	1	Yes
(2)	(1 2)	-1	No
(3)	(1 2 3)	1	Yes

To list all elements in A_3 we only need to list all permutations in S_3 with cycle shapes $\varnothing$ and (3). This is the identity plus all possible 3-cycles. Hence:

$$A_3 = \{e, (1\ 2\ 3), (1\ 3\ 2)\} = C_3.$$

Solution. (Question 7.4.4) The hint describes the method and works through the cases when the largest cycle is length 1 and when the largest cycle has length 2.

If the largest cycle has length 3, then the next largest might have length 1 or 2 but never 3 (because we are in A_5 so there are only 5 things to permute). Thus we have $(* \ * \ *)(*) *$ or $(* \ * \ *)(* \, *)$ and so it's easy to see the cycle shapes are (3) and $(3, 2)$.

If the largest cycle has length 4, then we have $(* \ * \ * \ *) \ *$ so the only possibility is (4). If the largest cycle has length 5, then we have $(* \ * \ * \ * \ *)$ so the only possible cycle shape is (5). Hence, we have the following table.

Cycle shape	Example of this cycle shape	Signature	In A_5?
$\varnothing$	e	1	Yes
(2)	$(1\ 2)$	-1	No
$(2, 2)$	$(1\ 2)(3\ 4)$	1	Yes
(3)	$(1\ 2\ 3)$	1	Yes
$(3, 2)$	$(1\ 2\ 3)(4\ 5)$	-1	No
(4)	$(1\ 2\ 3\ 4)$	-1	No
(5)	$(1\ 2\ 3\ 4\ 5)$	1	Yes

Solution. (Question 7.4.5) Here is a list of all elements, broken down by cycle shape. We know (or will soon see in lectures) that $|A_n| = |S_n|/2 = n!/2$, so $|A_5| = 60$.

Cycle shape $\varnothing$:
(There is just one permutation with this shape)

$$e$$

Cycle shape (2): Not in A_5

Cycle shape $(2)(2)$:
(Choose the element to be fixed first. There are $5 \cdot 3 = 15$ permutations with this shape).

[1 fixed]	$(2\ 3)(4\ 5)$,	$(2\ 4)(3\ 5)$,	$(2\ 5)(3\ 4)$
[2 fixed]	$(1\ 3)(4\ 5)$,	$(1\ 4)(3\ 5)$,	$(1\ 5)(3\ 4)$
[3 fixed]	$(1\ 2)(4\ 5)$,	$(1\ 4)(2\ 5)$,	$(1\ 5)(2\ 4)$
[4 fixed]	$(1\ 2)(3\ 5)$,	$(1\ 3)(2\ 5)$,	$(1\ 5)(2\ 4)$
[5 fixed]	$(1\ 2)(3\ 4)$,	$(1\ 3)(2\ 4)$,	$(1\ 4)(2\ 3)$.

Cycle shape (3):
(There are $2 \cdot (5 \cdot 4/2) = 20$ permutations with this shape)

$(1\ 2\ 3)$ & $(1\ 3\ 2)$, $(1\ 2\ 4)$ & $(1\ 4\ 2)$, $(1\ 2\ 5)$ & $(1\ 5\ 2)$,
$(1\ 3\ 4)$ & $(1\ 4\ 3)$, $(1\ 3\ 5)$ & $(1\ 5\ 3)$,
$(1\ 4\ 5)$ & $(1\ 5\ 4)$,
$(2\ 3\ 4)$ & $(2\ 4\ 3)$, $(2\ 3\ 5)$ & $(2\ 5\ 3)$,
$(2\ 4\ 5)$ & $(2\ 5\ 4)$,
$(3\ 4\ 5)$ & $(3\ 5\ 4)$.

Cycle shape $(3, 2)$: Not in A_5.

Cycle shape (4): Not in A_5.

Cycle shape (5):
(There are 5!/5 = 24 permutations with this shape)

$$\begin{array}{cccccc}
(1\ 2\ 3\ 4\ 5), & (1\ 2\ 3\ 5\ 4), & (1\ 2\ 4\ 3\ 5), & (1\ 2\ 4\ 5\ 3), & (1\ 2\ 5\ 3\ 4), & (1\ 2\ 5\ 4\ 3) \\
(1\ 3\ 2\ 4\ 5), & (1\ 3\ 2\ 5\ 4), & (1\ 3\ 4\ 2\ 5), & (1\ 3\ 4\ 5\ 2), & (1\ 3\ 5\ 2\ 4), & (1\ 3\ 5\ 4\ 2) \\
(1\ 4\ 3\ 2\ 5), & (1\ 4\ 3\ 5\ 2), & (1\ 4\ 2\ 3\ 5), & (1\ 4\ 2\ 5\ 3), & (1\ 4\ 5\ 3\ 2), & (1\ 4\ 5\ 2\ 3) \\
(1\ 5\ 3\ 4\ 2), & (1\ 5\ 3\ 2\ 4), & (1\ 5\ 4\ 3\ 2), & (1\ 5\ 4\ 2\ 3), & (1\ 5\ 2\ 3\ 4), & (1\ 5\ 2\ 4\ 3)
\end{array}$$

These are the 60 elements in A_5.