7.4 Exercises 7- Exercises on the alternating group

(S)

Question 7.4.1. Recall from lectures that the signature function $\sigma : S_n \to \{1, -1\}$ is used to define the alternating group A_n.

Calculate the signatures of the following permutations in S_{10}, and hence determine whether or not they lie in the alternating group A_{10}.

(a) $g = (6\ 1\ 4)(5\ 3\ 2\ 7)(8\ 9\ 10)$

(b) $h = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$

(c) g^{-1}

(d) gh

(e) $g^{-1}hg$

Question 7.4.2. Determine whether or not permutations with the following cycle shapes lie in an alternating group.

(a) Cycle shape $(3, 2, 2, 2)$

(b) Cycle shape $(7, 3, 3, 2, 2)$

(c) Cycle shape $(5, 5, 5)$

(d) Cycle shape $(2, 2, 2, 2, 2)$

Question 7.4.3. Without looking at your notes, try to list all the elements in A_3 by first listing all the cycle shapes that can be found in A_3.

Question 7.4.4. List all the cycles shapes that can be found in S_5 and determine which lie in A_5.

Question 7.4.5. Use your answer to Question 7.4.4 to list all elements of A_5.

Hint. (Question 7.4.1) In lectures we saw a corollary that allows us to easily calculate the signature of any permutation once it is written as a product of disjoint cycles. Also, the signature function is a homomorphism. These two things make the calculations very easy to compute.

Hint. (Question 7.4.2) All you need to know in order to work out the signature of a permutation is its cycle shape. And a permutation lies in an alternating group if and only if its signature is 1.

Hint. (Question 7.4.3) We covered this in lectures, but try not to look! Instead, think of all the possible cycle shapes in S_3, then calculate their signatures to work out those that can be found in A_3. For the cycle shapes that lie in A_3, write down all permutations with that cycle shape.

Hint. (Question 7.4.4) Think of any permutation in A_5, written as a product of disjoint cycles **including cycles of length 1**, with the cycles ordered according to length. With the brackets $()()\cdots()$ removed it will look like:

$$* \ * \ * \ * \ *$$

where the $*$ are the numbers $1, 2, 3, 4, 5$ in some order. Now put in the brackets, starting with the largest cycle. Note that the largest cycle can have length 1, 2, 3, 4 or 5.

If the largest cycle has length 1, then the permutation is the identity and has cycle shape $\varnothing$ so there's nothing to do here.

If the largest cycle has length 2, then the permutation looks like:

$$(* \ *) \ * \ * \ *$$

What options are there for the rest of the permutation? Well all remaining cycles must have length at most 2 because we decided that the largest cycle has length 2.
If the second cycle has length 1, then the permutation looks like $(* \ *) \ (*) \ * \ *$. Since all remaining cycles must now have length at most 1, we see that the permutation is $(* \ *) \ (*) \ (*) \ (*)$. The cycle shape is thus (2).
If the second cycle instead has length 2, then the permutation looks like $(* \ *) \ (* \ *) \ *$. We are left with only one option now, which is $(* \ *) \ (* \ *) \ (*)$. Hence the cycle shape of the permutation is $(2, 2)$.

Now work through the possibilities when the largest cycle length is 3, 4 or 5.

Hint. (Question 7.4.5) You already know all the possible cycle shapes in A_5 by Question 7.4.4. So, for each cycle shape in A_5, just write down all the permutations with that cycle shape. Just be careful not to double-count — remember for example that $(1 \ 2 \ 3) = (3 \ 1 \ 2) = (2 \ 3 \ 1)$.

There should be $|S_5|/2 = 5!/2 = 60$ permutations. Try to be very systematic about how you list the elements!