13.8 Solutions to Exercises 8 - Exercises on group actions

Solution. (Question 8.4.1)

Proof that the regular action is a group action. We just check that the two parts of the group action definition hold.
(1): for all $x \in X = G$ we have $\lambda(e)x = ex = x$. (2): for all $f, g \in G$ and all $x \in X = G$ we have $\lambda(fg)x = fgx = f(gx) = \lambda(f)(\lambda(g)x)$. □

Solution. (Question 8.4.2)

Proof that conjugation is a group action. We just check that the two parts of the group action definition hold.
(1): for all $x \in X = G$ we have $\lambda(e)x = e_G x e_G^{-1} = x$. (2): for all $f, g \in G$ and all $x \in X = G$ we have $\lambda(fg)x = (fg)x(fg)^{-1} = f(gxg^{-1})f^{-1} = \lambda(f)(\lambda(g)x)$. (This argument used the fact that: $(fg)^{-1} = g^{-1}f^{-1}$.) □

Solution. (Question 8.4.3) We must check that for all $g \in G$ and all $x \in X$ there is an element $\lambda(g)x \in X$ that satisfies:

(i) $\lambda(e)x = x$ for all $x \in X$; and

(ii) $\lambda(gh)x = \lambda(g)\left(\lambda(h)x\right)$ for all $g, h \in G$ and all $x \in X$.

Firstly, note that for all $g \in G$ and all $\{i, j\} \in X$ we have that $\lambda(g)\{i, j\} = \{gi, gj\} \in X$.
Secondly, $e \in S_6$ fixes every element of $\{1, 2, \ldots, 6\}$. Therefore, for all $\{i, j\} \in X$ we have $\lambda(e)\{i, j\} = \{ei, ej\} = \{i, j\}$. Hence (i) holds.
Thirdly, fix $g, h \in G = S_6$ and $\{i, j\} \in X$. Then $\lambda(gh)\{i, j\} = \{ghi, ghj\} = \{g(hi), g(hj)\} = \lambda(g)\{hi, hj\} = \lambda(g)\left(\lambda(h)\{i, j\}\right)$. Hence (ii) holds.
Since there is an action of G on X we know that X is (by definition) a G-set.

Solution. (Question 8.4.4) This is not an action. Recall the definition of an action: An *action* of G on X is a function λ such that for all $g \in G$ and all $x \in X$ there is an element $\lambda(g)x \in X$ that satisfies (i) and (ii) given above in the solution to Question 8.4.3.
The problem with the map λ we have been given for $(\mathbb{Z}, +)$ is that there is not always an element $\lambda(g)x \in X$. Indeed, take $z = -5 \in G$ and $n = 1 \in X$. Then according to the definition of λ we have $\lambda(z)n = z + n = -5 + 1 = -4 \notin X$.

Solution. (Question 8.4.5) Now for the regular action $G = S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ and $X = G$. Also,

$$\begin{aligned}
\lambda(e)(1\ 2\ 3) &= e(1\ 2\ 3) = (1\ 2\ 3) \\
\lambda((1\ 2))(1\ 2\ 3) &= (1\ 2)(1\ 2\ 3) = (2\ 3) \\
\lambda((1\ 3))(1\ 2\ 3) &= (1\ 3)(1\ 2\ 3) = (1\ 2) \\
\lambda((2\ 3))(1\ 2\ 3) &= (2\ 3)(1\ 2\ 3) = (1\ 3) \\
\lambda((1\ 2\ 3))(1\ 2\ 3) &= (1\ 2\ 3)(1\ 2\ 3) = (1\ 3\ 2) \\
\lambda((1\ 3\ 2))(1\ 2\ 3) &= (1\ 3\ 2)(1\ 2\ 3) = e
\end{aligned}$$

Therefore, the orbit of $(1\ 2\ 3)$ is $G(1\ 2\ 3) = \{(1\ 2\ 3), (2\ 3), (1\ 2), (1\ 3), (1\ 3\ 2), e\} = X$, so the action is transitive.

The stabiliser of $x = (1\ 2\ 3)$ is $\mathrm{Stab}_G(x) = \{g \in G : \lambda(g)x = x\}$. Looking at the above list, we see that the only element of S_3 that fixes $(1\ 2\ 3)$ is e, so $\mathrm{Stab}_G(x) = \langle e \rangle$.

In fact, this is always true for the regular action of any group on itself: the action is transitive and every stabiliser is $\langle e \rangle$. This is not difficult to prove, and you might enjoy trying to prove this fact yourself before we see it in lectures.

Solution. (Question 8.4.6) In this action we have that $G = S_3$ and $X = S_3$, and for all $g \in G$ and all $h \in X$,

$$\lambda(g)h = ghg^{-1}.$$

Again we note that $S_3 = \{e, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ and so we can calculate the orbit $G(1\ 2\ 3)$ under this action directly:

$$\begin{aligned}
\lambda(e)(1\ 2\ 3) &= e(1\ 2\ 3)e^{-1} = e(1\ 2\ 3)e = (1\ 2\ 3)\\
\lambda((1\ 2))(1\ 2\ 3) &= (1\ 2)(1\ 2\ 3)(1\ 2)^{-1} = (1\ 2)(1\ 2\ 3)(21) = (1\ 3\ 2)\\
\lambda((1\ 3))(1\ 2\ 3) &= (1\ 3)(1\ 2\ 3)(1\ 3)^{-1} = (1\ 3)(1\ 2\ 3)(3\ 1) = (1\ 3\ 2)\\
\lambda((2\ 3))(1\ 2\ 3) &= (2\ 3)(1\ 2\ 3)(2\ 3)^{-1} = (2\ 3)(1\ 2\ 3)(3\ 2) = (1\ 3\ 2)\\
\lambda((1\ 2\ 3))(1\ 2\ 3) &= (1\ 2\ 3)(1\ 2\ 3)(1\ 2\ 3)^{-1} = (1\ 2\ 3)(1\ 2\ 3)(3\ 2\ 1) = (1\ 2\ 3)\\
\lambda((1\ 3\ 2))(1\ 2\ 3) &= (1\ 3\ 2)(1\ 2\ 3)(1\ 3\ 2)^{-1} = (1\ 3\ 2)(1\ 2\ 3)(2\ 3\ 1) = (1\ 2\ 3)
\end{aligned}$$

Hence the orbit $G(1\ 2\ 3)$ is $\{(1\ 2\ 3), (1\ 3\ 2)\} \neq X$. Since the orbit is smaller than X, we see that the action is not transitive.

The stabiliser of $x = (1\ 2\ 3) \in X$ is $\mathrm{Stab}_G(x) = \{g \in G : \lambda(g)x = x\} = \{g \in S_3 : g(1\ 2\ 3)g^{-1} = (1\ 2\ 3)\}$. We can see from the above list which elements of S_3 lie in the stabiliser. When we do this we see that,

$$\mathrm{Stab}_G(x) = \{e, (1\ 2\ 3), (1\ 3\ 2)\}.$$

Solution. (Question 8.4.7) This is an easy proof using the Quick Subgroup Test.

- Identity: by definition we know that $\lambda(e)x = x$, so $x \in \mathrm{Stab}_G(x)$.
- Closure: if $g, h \in \mathrm{Stab}_G(x)$ then $\lambda(gh)x = \lambda(g)\lambda(h)x = \lambda(g)x = x$. Hence $gh \in \mathrm{Stab}_G(x)$.
- Inverse: if $g \in \mathrm{Stab}_G(x)$, then $\lambda(g)x = x$, and so because λ is an action we know that $x = \lambda(e)x = \lambda(g^{-1}g)x = \lambda(g^{-1})(\lambda(g)x) = \lambda(g^{-1})x$. Hence $g^{-1} \in \mathrm{Stab}_G(x)$.

Solution. (Question 8.4.8) Let $\lambda : G \to \mathrm{Sym}(X)$ be a homomorphism. We must check that for all $g \in G$ and all $x \in X$ there is an element $\lambda(g)x \in X$ that satisfies:

(i) $\lambda(e)x = x$ for all $x \in X$; and

(ii) $\lambda(gh)x = \lambda(g)\left(\lambda(h)x\right)$ for all $g, h \in G$ and all $x \in X$.

Now fix $g, h \in G$ and $x \in X$. Since $\lambda(g) \in \mathrm{Sym}(X)$ we know that $\lambda(g)x \in X$. Also, we saw in lectures that homomorphisms always map identity elements to identity elements, so $\lambda(e_G) = e$. Thus $\lambda(e_G)x = ex = x$, so (i) holds. Finally, because λ is a homomorphism we have $\lambda(gh) = \lambda(g)\lambda(h)$, and therefore $\lambda(gh)x = \lambda(g)\lambda(h)x = \lambda(g)\left(\lambda(h)x\right)$, so (ii) holds. Whence λ is an action of G on X.