8.4 Exercises 8- Exercises on group actions

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Question 8.4.1. Check that you agree that the regular action (see Example 8.1.7) satisfies the definition of a group action. Do this by checking the two parts (1) and (2) of the group action definition (Definition 8.1.1) hold.

Question 8.4.2. Check that you agree that the conjugation action (see Example 8.1.8) satisfies the definition of a group action. Do this by checking the two parts (1) and (2) of the group action definition (Definition 8.1.1) hold.

Question 8.4.3. Let $G = S_6$ and let

$$X = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\}\},$$
$$\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}.$$

Suppose $\lambda(g)\{i,j\} = \{gi, gj\}$ for all $\{i,j\} \in X$ and all $g \in G$. Prove that λ is an action of G on X and hence that X is a G-set.

Question 8.4.4. Let $G = (\mathbb{Z}, +)$, and let $X = \mathbb{N}$. Suppose $\lambda(z)n = z + n$ for all $z \in G$ and all $n \in X$. Is λ an action of G on X? Justify your answer.

Question 8.4.5. Let $G = S_3$ act on itself via the regular action. Calculate the orbit of $(1\ 2\ 3)$ and deduce whether or not the action is transitive. Also find the stabiliser of $(1\ 2\ 3)$ in this action.

Question 8.4.6. Let $G = S_3$ act on itself again, but this time via the (left) conjugation action. Calculate the orbit of $(1\ 2\ 3)$ and deduce whether or not the action is transitive. Also find the stabiliser of $(1\ 2\ 3)$ in this action.

Question 8.4.7. Let X be a G-set, with $x \in X$. Prove that $\mathrm{Stab}_G(x)$ is a subgroup of G.

Question 8.4.8. Let $\lambda : G \to \mathrm{Sym}(X)$ be a homomorphism. Prove that λ is an action of G on X.

[Note that in lectures we proved every action was a homomorphism into $\mathrm{Sym}(X)$. Here you are proving that every homomorphism into $\mathrm{Sym}(X)$ is an action. Hence actions on X and homomorphisms into $\mathrm{Sym}(X)$ are the same thing.]

Hint. (Question 8.4.3) Look at the definition of a group action. There are just two conditions that you need to check. Now check that they hold!

Hint. (Question 8.4.4) Look carefully at the definition of an action. There is more to the definition than just the two conditions.

Hint. (Question 8.4.5) This question is straightforward when you remember the definition of the regular action. Recall that the orbit of $(1\ 2\ 3)$ is everything $(1\ 2\ 3)$ gets mapped to by the action, so $\{\lambda(g)(1\ 2\ 3) : g \in S_3\}$.

Hint. (Question 8.4.6) This question is straightforward when you remember the definition of the conjugation action.

Hint. (Question 8.4.7) Use the Quick Subgroup Test.

Hint. (Question 8.4.8) Just check that λ satisfies the conditions of being a group action.