

## 13.9 Solutions to Exercises 9 - Exercises on the orbit stabiliser theorem and the orbit counting theorem

**Solution.** (Question 9.5.1) A set of orbit representatives is  $\{1, 4, 7\}$ . This is because:

$$G1 = \{1, 2, 3\}, \quad G4 = \{4, 5, 6\}, \quad \text{and} \quad G7 = \{7, 8, 9\}.$$

These are all disjoint, so  $X = G1 \cup G4 \cup G7$  is a way of writing  $X$  as a disjoint union of orbits. Of course you might also write  $X = G2 \cup G4 \cup G8$ , etc.

[You definitely can't write  $G1 \cup G2 \cup G4 \cup G8$  because  $G1$  and  $G2$  are not disjoint.]

**Solution.** (Question 9.5.2) This is an easy application of the Orbit-Stabiliser Theorem, which states that for all  $x$  in a  $G$ -set  $X$  we have,

$$|Gx| = [G : \text{Stab}_G(x)].$$

When  $G$  is finite we can do even better:

$$|Gx| = |G| / |\text{Stab}_G(x)|.$$

- (a) Here  $|Gx| = 4$  and  $|G| = 20$ . Plugging in gives  $|\text{Stab}_G(x)| = 5$ .
- (b) Since  $|G| = 20$  and  $|\text{Stab}_G(y)| = 2$ , we have  $|Gy| = 10$ .

**Solution.** (Question 9.5.3) If you draw out a pentagon and label its corners, you will see that the orbit under  $D_{10}$  of the corner 1 is  $\{1, 2, 3, 4, 5\}$  so the action is transitive. We also have that the stabiliser of corner 1 in  $D_{10}$  is  $\{e, (2\ 5)(3\ 4)\}$ . Plugging all this into the Orbit-Stabiliser Theorem gives,

$$|D_{10}1| = |\{1, 2, 3, 4, 5\}| = 5 = \frac{10}{2} = \frac{|D_{10}|}{|\text{Stab}_{D_{10}}(1)|} = [D_{10} : \text{Stab}_{D_{10}}(1)].$$

Thus, the Orbit-Stabiliser Theorem holds in this case.

**Solution.** (Question 9.5.4) Suppose  $Y$  is an orbit and  $x \in Y$ . Since  $Y$  is an orbit we have  $Y = Gy$  for some  $y \in X$ . Therefore  $x = \lambda(g)y$  for some  $g \in G$ . This implies that  $y = \lambda(g^{-1})x$ .

Note that for all  $h \in G$  we have  $\lambda(h)x = \lambda(h)\lambda(g)y = \lambda(hg)y \in Gy$ . Hence  $Gx \subseteq Gy$ .

On the other hand, for all  $h \in G$  we have  $\lambda(h)y = \lambda(h)\lambda(g^{-1})x = \lambda(hg^{-1})x \in Gx$ . Hence  $Gy \subseteq Gx$ .

Therefore,  $Gx = Gy = Y$ .

**Solution.** (Question 9.5.5) The elements in  $A_7$  that fix 1 are the elements of the stabiliser  $\text{Stab}_{A_7}(1)$ . So, this question requires just an easy application of the Orbit-Stabiliser Theorem. We know (because  $A_7$  is transitive) that the orbit  $A_71$  is equal to  $X$ . Thus,

$$7 = |A_71| = |A_7| / |\text{Stab}_{A_7}(1)| = \frac{7!/2}{|\text{Stab}_{A_7}(1)|}.$$

Rearranging gives  $|\text{Stab}_{A_7}(1)| = (7!/2)/7 = 360$ .

**Solution.** (Question 9.5.6) Fix  $x, y \in X$ . Since  $G$  acts transitively on  $X$ , we know that  $Gx = X = Gy$ , and so importantly  $|Gx| = |Gy|$ . Since  $G$  is a finite group, we know by the Orbit-Stabiliser Theorem that,

$$\frac{|G|}{|\text{Stab}_G(x)|} = |Gx| = |Gy| = \frac{|G|}{|\text{Stab}_G(y)|}.$$

Therefore,  $|\text{Stab}_G(x)| = |\text{Stab}_G(y)|$ .

**Solution.** (Question 9.5.7) We list all elements in  $S_4$  and the elements of  $Y$  that they fix.

$g \in S_4$	$\text{Fix}_Y(g)$	$ \text{Fix}_Y(g) $
$e$	$Y$	16
$(1\ 2)$	$\{(i, j) : i, j \in \{3, 4\}\}$	4
$(1\ 3)$	$\{(i, j) : i, j \in \{2, 4\}\}$	4
$(1\ 4)$	$\{(i, j) : i, j \in \{2, 3\}\}$	4
$(2\ 3)$	$\{(i, j) : i, j \in \{1, 4\}\}$	4
$(2\ 4)$	$\{(i, j) : i, j \in \{1, 3\}\}$	4
$(3\ 4)$	$\{(i, j) : i, j \in \{1, 2\}\}$	4
$(1\ 2\ 3)$	$\{(4, 4)\}$	1
$(1\ 3\ 2)$	$\{(4, 4)\}$	1
$(1\ 2\ 4)$	$\{(3, 3)\}$	1
$(1\ 4\ 2)$	$\{(3, 3)\}$	1
$(1\ 3\ 4)$	$\{(2, 2)\}$	1
$(1\ 4\ 3)$	$\{(2, 2)\}$	1
$(2\ 3\ 4)$	$\{(1, 1)\}$	1
$(2\ 4\ 3)$	$\{(1, 1)\}$	1
$(1\ 2\ 3\ 4)$	$\emptyset$	0
$(1\ 3)(2\ 4)$	$\emptyset$	0
$(1\ 4\ 3\ 2)$	$\emptyset$	0
$(1\ 2\ 4\ 3)$	$\emptyset$	0
$(1\ 4)(2\ 3)$	$\emptyset$	0
$(1\ 3\ 4\ 2)$	$\emptyset$	0
$(1\ 3\ 2\ 4)$	$\emptyset$	0
$(1\ 2)(3\ 4)$	$\emptyset$	0
$(1\ 4\ 2\ 3)$	$\emptyset$	0

Now apply the Orbit Counting Theorem: the number of orbits is  $\frac{1}{24}(16 + 4 \cdot 6 + 1 \cdot 8) = 2$ .

**Solution.** (Question 9.5.8) Recall that the regular action is  $\lambda(g)h = gh$  for all  $g \in G$  and for all  $h \in G$ .

Now for all  $g \in G$  we have that  $h \in \text{Fix}_G(g) \iff \lambda(g)h = h \iff gh = h \iff g = e_G$ . Therefore, if  $g \neq e_G$  then  $\text{Fix}_G(g) = \emptyset$ . Of course  $\text{Fix}_G(e_G) = G$ . Applying the Orbit Counting Theorem we therefore have,

$$\# \text{ orbits} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}_G(g)| = \frac{1}{|G|} (|G| + 0 + \Pi + 0) = \frac{|G|}{|G|} = 1.$$

Thus, there is only one orbits so the action is transitive.

**Solution.** (Question 9.5.9) As noted in the hint, this question is equivalent to asking: how many octagons with corners coloured white/black are there with precisely 3 corners coloured black, if we consider two such octagons to be the same if one can be transformed into the other by an element of  $D_{16}$ .

Using the analysis we saw in lectures, we let  $Y$  be the set of all octagons whose corners are coloured white/black in such a way that precisely 3 corners on each octagon are black. Then  $|Y| = \binom{8}{3} = 56$ .

We need a sensible way of thinking about the elements of  $Y$ , so let's list the corner colours (starting from the top and working anticlockwise) as an 8-tuple. For example, the octagon shown in the hint for this question is  $(b, b, w, w, w, w, b, w)$ .

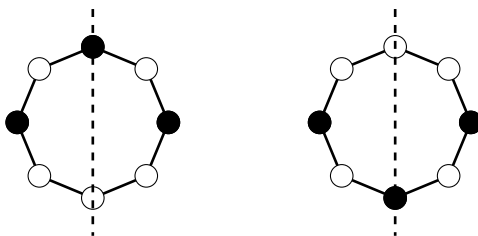
Now  $D_{16}$  acts on  $Y$  in the obvious way: the rotation  $\rho$  (by  $360/8$  degrees anticlockwise) sends  $(b, b, w, w, w, w, b, w)$  to  $(w, b, b, w, w, w, b, b)$ , and so on.

We first note that  $\rho, \rho^2, \dots, \rho^7$  all fix no element in  $Y$  (you should draw some diagrams to convince yourself this is true). Thus,

$$|\text{Fix}_Y(\rho^i)| = 0 \quad \text{for } i = 1, 2, \dots, 7.$$

As noted in lectures,  $D_{16}$  contains two types of reflections: four whose line of symmetry is through a corner (let's call these  $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ ), and four whose line of symmetry is through an edge (let's call these  $\sigma_5, \sigma_6, \sigma_7, \sigma_8$ ).

Consider  $\sigma_1$  and some element  $y$  of  $Y$ . If the two corners that the line of symmetry passes through are both white or both black, then  $y$  cannot be fixed by  $\sigma_1$  (because one side of the line of symmetry has too many black corners). On the other hand, if precisely one of the corners that the line of symmetry passes through is black, then  $y$  can be fixed if the remaining two black corners are opposite each other (the diagram below will help you picture this).



Hence  $|\text{Fix}_Y(\sigma_1)| = 6$  (there are three elements of  $Y$  that are fixed for each of the two corners lying on the line of symmetry). The reflections  $\sigma_2, \sigma_3, \sigma_4$  are similar. Thus,

$$|\text{Fix}_Y(\sigma_i)| = 6 \quad \text{for } i = 1, 2, 3, 4.$$

Finally, we note that the reflections  $\sigma_5, \sigma_6, \sigma_7, \sigma_8$  can never fix an element of  $Y$ , because their lines of symmetry will always divide the octagon into a half with two black corners, and a half with one black corner. Thus,

$$|\text{Fix}_Y(\sigma_i)| = 0 \quad \text{for } i = 5, 6, 7, 8.$$

Of course we also have that  $\text{Fix}_Y(e) = Y$ , so  $|\text{Fix}_Y(e)| = 56$ . Plugging all this into the Orbit Counting Theorem, we have,

$$\# \text{ orbits} = \frac{1}{16}(56 + 0 \cdot 7 + 6 \cdot 4 + 0 \cdot 4) = 5.$$

Hence, the number of distinct necklaces that can be formed using precisely 3 black beads and 5 white beads is 5.

**Solution.** (Question 9.5.10) Suppose  $Gx \cap Gy \neq \emptyset$ . We will prove that  $Gx = Gy$ . Since  $Gx \cap Gy \neq \emptyset$ , we can find  $z \in Gx \cap Gy$ . Hence there is  $g_1, g_2 \in G$  such that

$$\lambda(g_1)x = z = \lambda(g_2)y.$$

Since  $\lambda$  is a homomorphism,  $\lambda(g_i)^{-1} = \lambda(g_i^{-1})$  for  $i = 1, 2$ . Therefore  $y = \lambda(g_2^{-1}g_1)x \in Gx$  and  $x = \lambda(g_1^{-1}g_2)y \in Gy$ . Since  $y \in Gx$  we have  $Gy \subseteq Gx$ , and since  $x \in Gy$  we have  $Gx \subseteq Gy$ . Hence  $Gx = Gy$ .