9.5 Exercises 9- Exercises on the orbit stabiliser theorem and the orbit counting theorem

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Question 9.5.1. Consider the cyclic group $G = \langle (1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9) \rangle \leq S_9$. It acts on the set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in the usual way. Find a set of orbit representatives (i.e. one element from each orbit) for this action, and write X as a disjoint union of orbits.

Question 9.5.2. Suppose G is a group of order 20, and G acts on a set X.

(a) If $x \in X$ and the orbit Gx has size 4, what is $|\mathrm{Stab}_G(x)|$?

(b) If $y \in X$ is such that $|\mathrm{Stab}_G(y)| = 2$, what is the size of the orbit of y?

Question 9.5.3. Recall that D_{10} acts as a group of symmetries of the regular pentagon. Find the orbit of the corner labelled 1 and find its stabiliser in D_{10}. Hence verify that the Orbit-Stabiliser Theorem holds in this situation.

Question 9.5.4. Let X be a G-set. Prove that if Y is an orbit of G acting on X, and x is any element in Y, then $Gx = Y$. [Note that this means that if x, y lie in the same orbit, then $Gx = Gy$.]

Question 9.5.5. For $n \geq 2$ the group A_7 acts transitively on the set $X = \{1, 2, 3, 4, 5, 6, 7\}$. How many elements in A_7 fix the number 1?

Question 9.5.6. Let G be a finite group and let X be a transitive G-set. Prove that, $|\mathrm{Stab}_G(x)| = |\mathrm{Stab}_G(y)|$ for all $x, y \in X$.

Question 9.5.7. The group S_4 acts on the set $Y = \{(i, j) : i, j \in \{1, 2, 3, 4\}\}$ via the action $\lambda(g)(i, j) = (gi, gj)$ for all $g \in S_4$ and all $(i, j) \in Y$. Use the Orbit Counting Theorem to determine the number of orbits of S_4 on Y. (There is an easier way to work out the number of orbits, but this question is about practicing the Orbit Counting Theorem).

Question 9.5.8. Let G be a group. We know that G acts on itself via the left regular action. Use the Orbit Counting Theorem to determine the number of orbits of this action. (There is an easier way to work out the number of orbits, but this question is about practicing the Orbit Counting Theorem).

Question 9.5.9. Use the Orbit Counting Theorem to answer the following question: How many different necklaces can be formed using precisely 3 black beads and 5 white beads (assuming the beads can pass freely over the clasp)?

Question 9.5.10. Let X be a G-set with action λ and $x, y \in X$. Prove that either $Gx = Gy$ or $Gx \cap Gy = \varnothing$.

Hint. (Question 9.5.1) Look up the definition of orbit representatives.

Hint. (Question 9.5.2) [Orbit Stabiliser Theorem] Use the Orbit Stabiliser Theorem.

Hint. (Question 9.5.3) [Orbit Stabiliser Theorem] Draw a pentagon and label the corners. Now write down the orbit, and the stabiliser. Finally, use the orbit stabiliser theorem.

Hint. (Question 9.5.4) [Orbit Stabiliser Theorem] It might help to note that $Y = Gy$ for some $y \in X$.

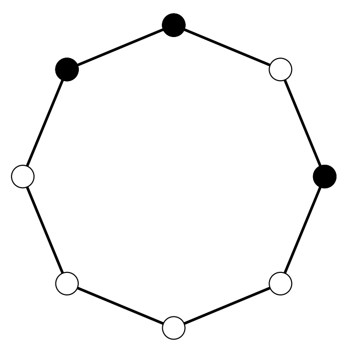
Hint. (Question 9.5.5) [Orbit Stabiliser Theorem] This question can be answered really quickly. If you find yourself trying to list all the elements in A_7 then you are trying way too hard.

Hint. (Question 9.5.6) [Orbit Stabiliser Theorem] Try using the Orbit Stabiliser Theorem.

Hint. (Question 9.5.7) [Orbit counting theorem] Try listing every element of S_4 in a table, and next to each element $g \in S_4$ record $\mathrm{Fix}_\mathrm{Y}(\mathrm{g})$.

Hint. (Question 9.5.8) [Orbit counting theorem] Recall the regular action: for all $g \in G$ and for all $h \in G$ we have $\lambda(g)h = gh$.

Hint. (Question 9.5.9) [Orbit counting theorem] For any such necklace you can arrange it to look like an octagon. For example:



Two necklaces N_1 and N_2 are the same if, when they are arranged like this, some element of D_{16} can transform one into the other.

Hint. (Question 9.5.10) [Neither orbit stabiliser theorem, nor orbit counting theorem] Suppose $Gx \cap Gy \neq \varnothing$. This implies there is some $z \in X$ such that $z \in Gx$ and $z \in Gy$. Can you now prove that it must then be the case that $Gx = Gy$? If not, try writing z as $\lambda(g_1)x$ and as $\lambda(g_2)y$ for some $g_1, g_2 \in G$.