

TODAY

- Recap/solution last session
- Numerical integration of an integral
- Solving initial value problems with second or higher order derivatives



RECAP AND SOLUTION OF SOME OF THE EXERCISES LAST WEEK



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STABILITY FORWARD EULER METHOD

The explicit (forward) Euler method is **conditionally stable**. It becomes **unstable** for $a\Delta t > 2$; the numerical solution then runs away from the actual solution.

The reason is that the numerical scheme is

$$y(t + \Delta t) = (1 - a\Delta t)y(t)$$

and hence

$$y(t + n\Delta t) = (1 - a\Delta t)^n y(t)$$

and this diverges when $n \rightarrow \infty$ for $|1 - a\Delta t| > 1$, which implies (since $a > 0$), $a\Delta t > 2$.



STABILITY BACKWARD EULER METHOD

The implicit (backward) Euler method is **unconditionally stable** for this stable ODE.

The reason is that the scheme is

$$y(t + \Delta t) = y(t)/(1 + a\Delta t)$$

and hence

$$y(t + n\Delta t) = y(t)/(1 + a\Delta t)^n$$

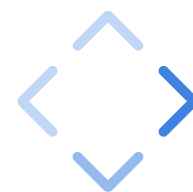
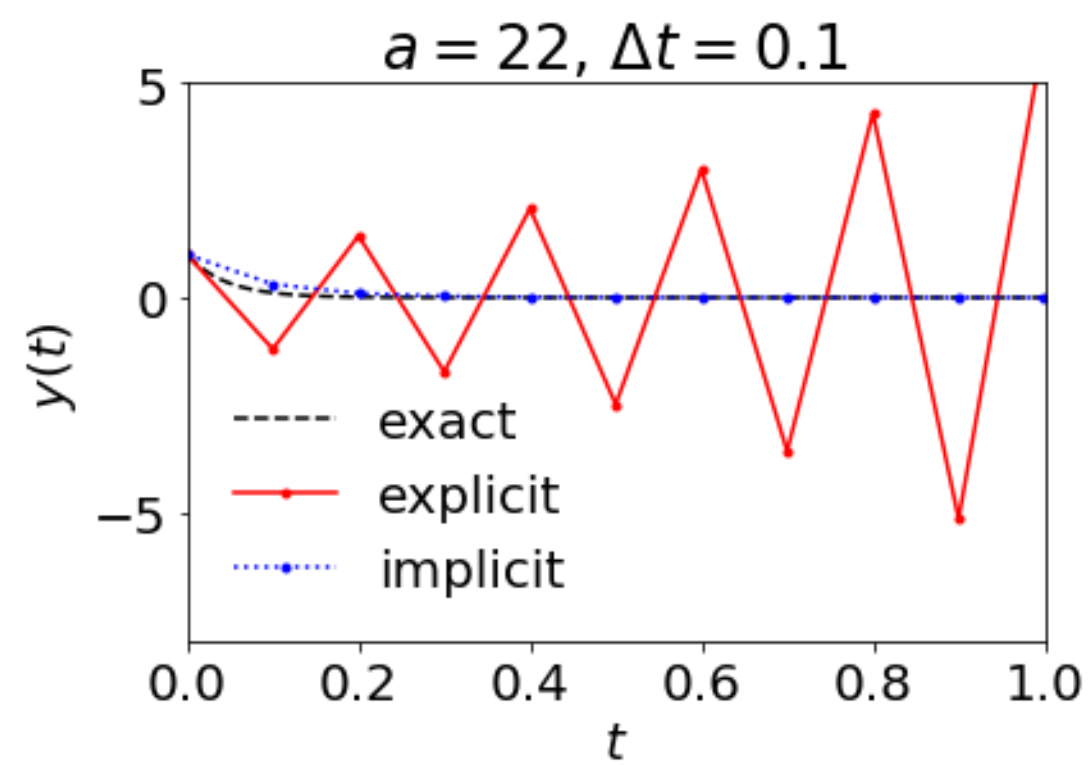
and since $a > 0$, $(1 + a\Delta t) > 1$, $1/(1 + a\Delta t) < 1$ and hence it always goes to zero for $n \rightarrow \infty$, irrespective of Δt .



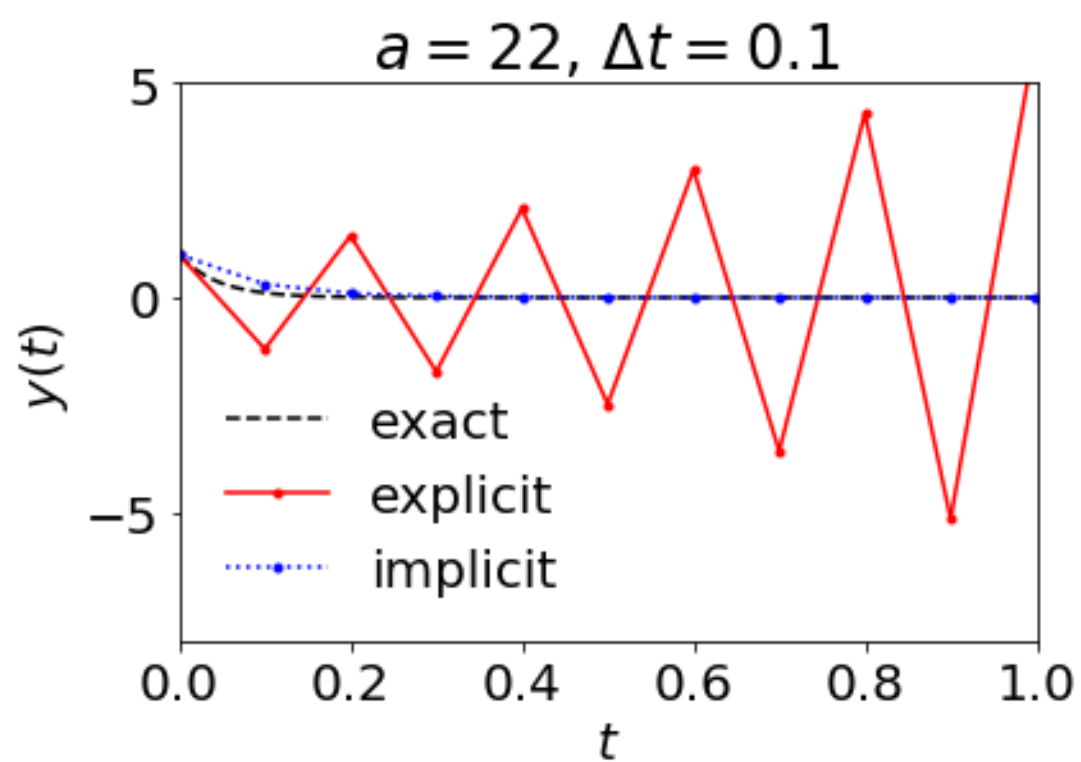
EXAMPLE GLOBAL ERROR



EXAMPLE GLOBAL ERROR



EXAMPLE GLOBAL ERROR



This shows that for $a \cdot \Delta t = 2.2 > 2$ the explicit Euler method is unstable (errors grow), but the implicit Euler method is stable (errors don't grow).



1.1

$$y = 4 - x$$

1.2

$$y = \frac{c - ax}{B}$$

1.3

$$y = \frac{(d - a)x + e}{b - c}$$

1.4

$$y_{\pm} = \frac{c \pm \sqrt{c^2 - 4b((a - d)x - e)}}{2b}$$



2

2.1

$$y_{i+1} = 4 - y_i$$

2.2

$$y_{i+1} = \frac{1}{1 + \Delta t a} y_i$$



3

$$\begin{aligned}\frac{z_{i+1} - z_i}{\Delta t} &= 4c_1 z_{i+1} - 8c_2(t + \Delta t)^2 \\ z_{i+1}(1 - 4c_1\Delta t) &= z_i - \Delta t 8c_2(t + \Delta t)^2 \\ z_{i+1} &= \frac{z_i - \Delta t 8c_2(t + \Delta t)^2}{1 - 4c_1\Delta t}\end{aligned}$$



NUMERICAL SOLUTION OF AN INTEGRAL

Integration:

$$F(z) = \int_a^z f(x)dx$$

How to determine $F(z)$ numerically?

One way to determine the integral is to solve the associated ODE, obtained by differentiating both sides:

$$\frac{dF(z)}{dz} = f(z), \text{ with } F(a) = 0$$

The initial condition follows from that the integral from $z = a$ to $z = a$ (the area) is zero.

We can use the previously discussed methods to find the integral. E.g., using forward/backward Euler (first order), (implicit) trapezoid, midpoint or Ralston's method (second order), or a higher order Runge-Kutta method such as RK4 (fourth order), by discretizing the interval.



EXAMPLE

$$y(t_{\max}) = \int_0^{t_{\max}} (-2t^3 + 12t^2 - 20t + 8.5) dt$$

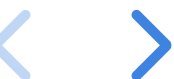
Assume $t_{\max} = 1$.

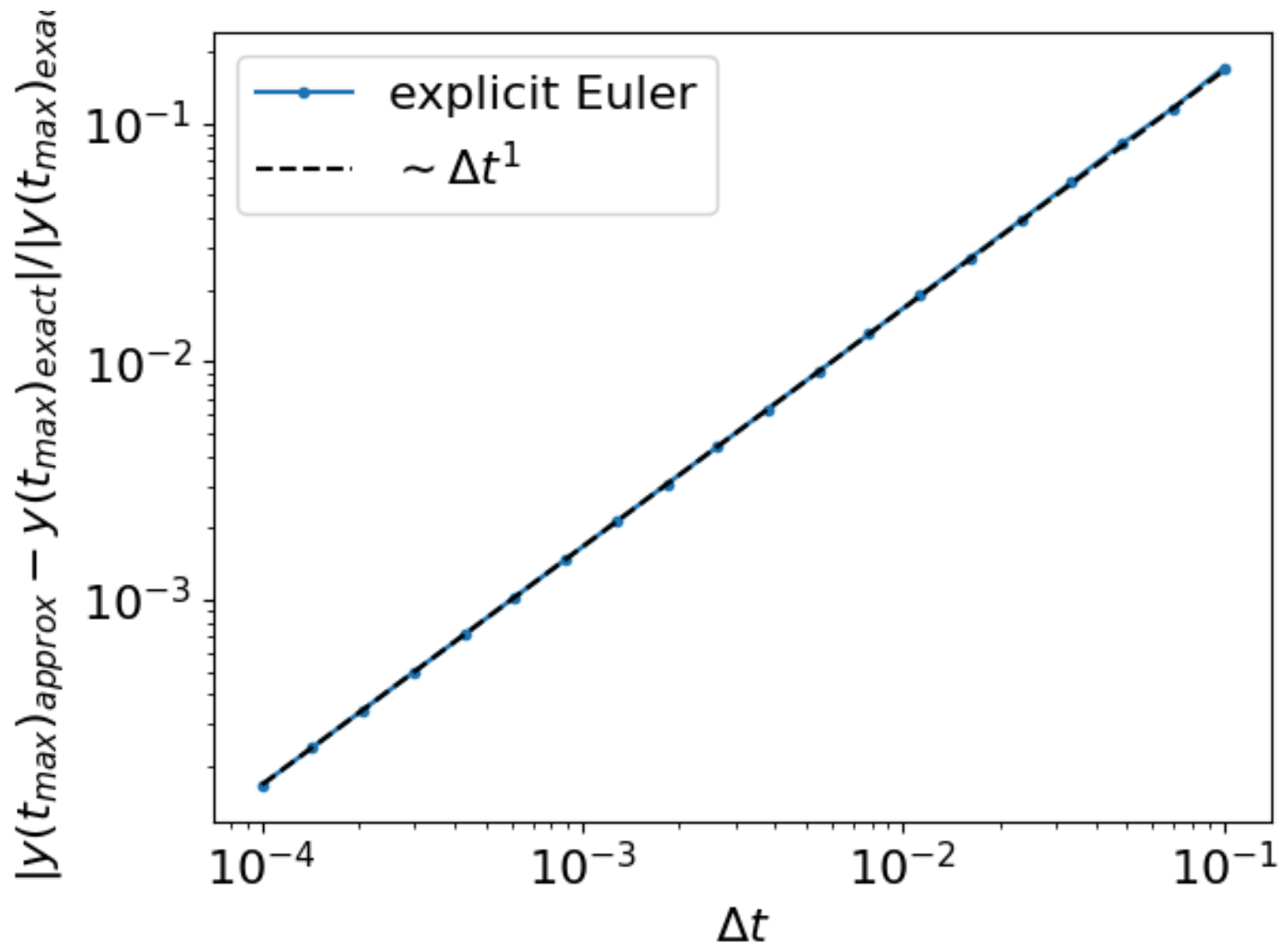
So the corresponding ODE is

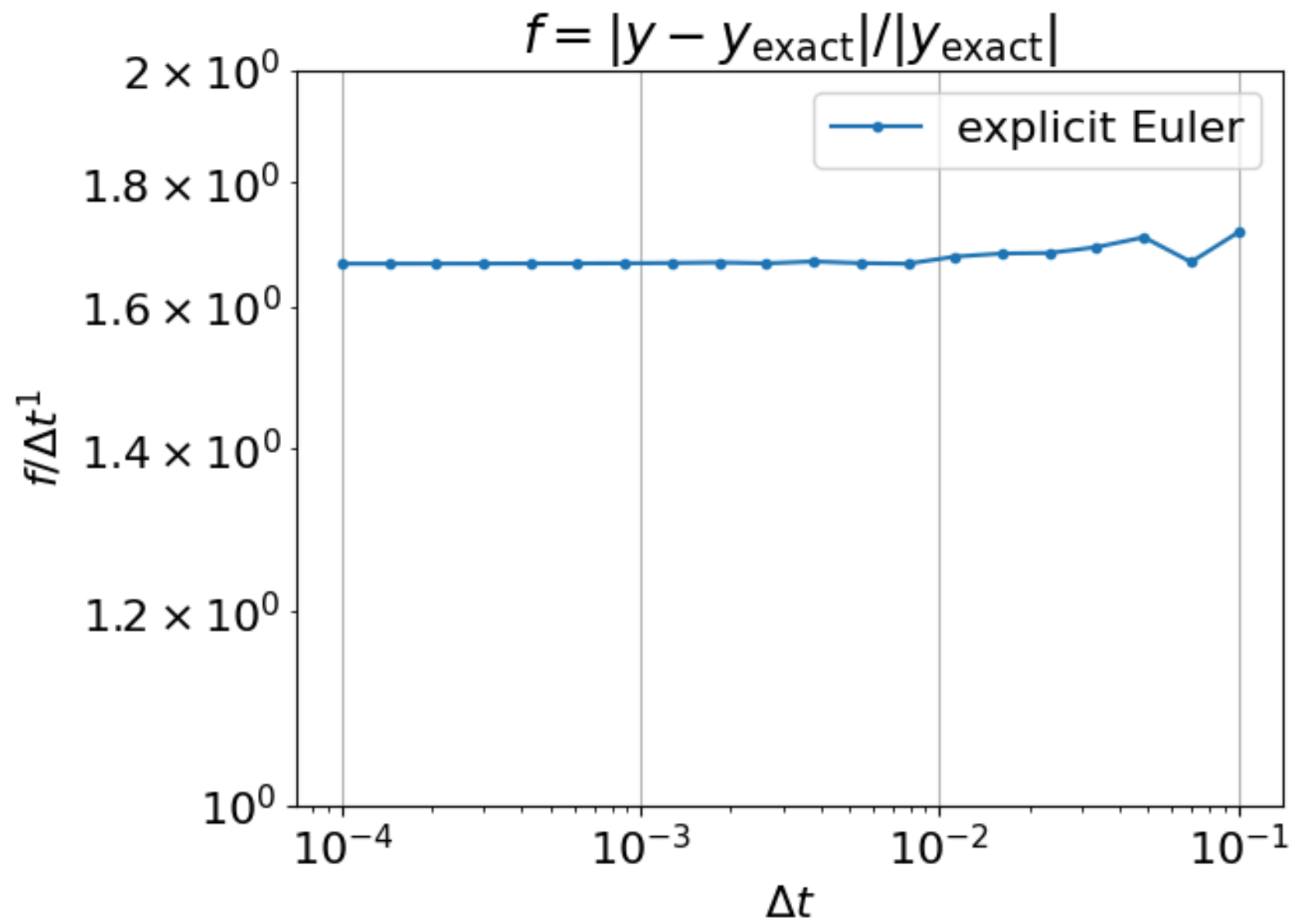
$$\frac{dy}{dt} = -2t^3 + 12t^2 - 20t + 8.5$$

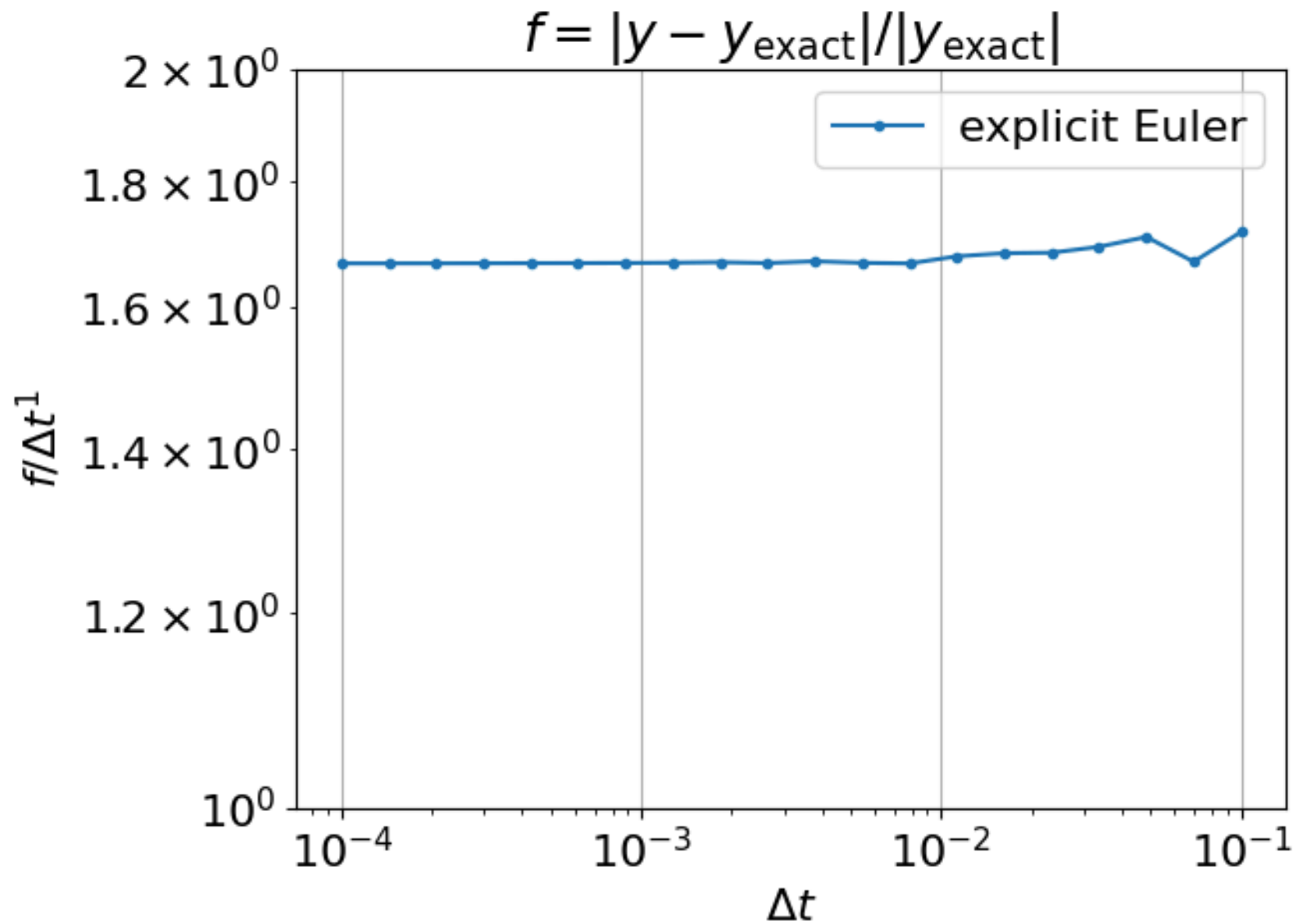
with $y(0) = 0$.

Since the integrand is just a polynomial, it is easy to compare with the analytical solution.









Previous slide: error vs Δt . When divided by Δt^1 (second graph) the result is a flat line for sufficiently small time step, so approximately a constant. The global error therefore is $O(\Delta t^1)$, which is indicative of a first order method.



HIGHER ORDER ODES

Up to now we only considered first order ODEs: only a first order derivative occurred -- no second or higher order derivatives.

How to numerically solve ODEs which have higher order derivatives?



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How to numerically solve ODEs which have higher order derivatives?

Solution: a *higher order ODE*, such as second order, can be rewritten as *a system of first order ODEs*.



EXAMPLE 2ND ORDER ODE

General example second order ODE

$$\frac{d^2y}{dt^2} = g\left(t, y, \frac{dy}{dt}\right)$$

Specific example second order ODE

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = (1 + 4t)\sqrt{y}$$

Therefore for this example $g(t, y, y') = (1 + 4t)\sqrt{y} - y'$.



HOW TO REDUCE THE ORDER?

For an ODE of order n ,

$$\frac{d^n y}{dt^n} = g(t, y, y', \dots, y^{(n-1)})$$

we can introduce new variables z_1, \dots, z_n . Take

$$z_1(t) = y(t)$$

$$z_2(t) = y(t)'$$

\vdots

$$z_n(t) = \frac{d^{n-1} y(t)}{dt^{n-1}}$$

So $z_m(t)$ corresponds to the $m - 1$ 'th order derivative of $y(t)$ for all $1 \leq m \leq n$. We have introduced a new variable for y and for each of its derivatives, *except* for the highest derivative.



Now differentiate both sides of the system of equations

$$\begin{aligned}z_1(t)' &= y(t)' \\z_2(t)' &= y(t)'' \\&\vdots \\z_n(t)' &= \frac{d^n y(t)}{dt^n}\end{aligned}$$

and substitute away the relations for $y(t)$ and its derivatives on the RHS. E.g., replace $y(t)'$ by $z_2(t)$.

Hence we have

$$\begin{aligned}z_1(t)' &= z_2(t) \\z_2(t)' &= z_3(t) \\&\vdots \\z_{n-1}(t)' &= z_n(t) \\z_n(t)' &= g(t, z_1(t), z_2(t), \dots, z_{n-1}(t))\end{aligned}$$

So all the new variables are given by *simple relations*, except for the last one, z_n' , which involves g .



EXAMPLE ORDER REDUCTION

Our example

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = (1 + 4t)\sqrt{y}$$

Let us first write it in the standard form by bringing the lower-order derivatives to the RHS:

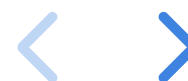
$$\frac{d^2y}{dt^2} = (1 + 4t)\sqrt{y} - \frac{dy}{dt}$$

Use the substitutions

$$\begin{aligned}z_1(t) &= y(t) \\z_2(t) &= y(t)'\end{aligned}$$

This leads to

$$\begin{aligned}z_1(t)' &= y(t)' = z_2(t) \\z_2(t)' &= y(t)'' = (1 + 4t)\sqrt{y} - y' = (1 + 4t)\sqrt{z_1(t)} - z_2(t)\end{aligned}$$



So we have the equivalent coupled system of first order ODEs

$$\begin{aligned}z_1(t)' &= z_2(t) \\z_2(t)' &= (1 + 4t)\sqrt{z_1(t)} - z_2(t)\end{aligned}$$

These can be written, in a similar fashion to last week:

$$\begin{aligned}z_1(t)' &= g_1(t, z_1(t), z_2(t)) \\z_2(t)' &= g_2(t, z_1(t), z_2(t))\end{aligned}$$

with $g_1(t, z_1(t), z_2(t)) = z_2(t)$ and $g_2(t, z_1(t), z_2(t)) = (1 + 4t)\sqrt{z_1(t)} - z_2(t)$.

Hence we can use all the explicit methods such as forward Euler, just as with last week.



EXERCISES SESSION 5

See assignment on Blackboard.

