Tensor Analysis– Main Exam's Answers

Question 1

(a) Compute the outer product $M \otimes N$ of the following two matrices: [9 marks]

$$M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and } N = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}.$$

(b) Suppose $T_{ijk\ell}$ is a rank-four tensor. Using the transformation rule, show that T_{ijij} is a scalar. [8 marks]

(c) Using suffix notation, find an expression involving no cross products for [8 marks]

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}).$$

Answers:

(a)

$$M \otimes N = \begin{pmatrix} 1 & 0 & 3 & 2 & 0 & 6 \\ 0 & 1 & -1 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

(c) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{z}) - (\mathbf{u} \cdot \mathbf{z})(\mathbf{v} \cdot \mathbf{w}).$

Question 2

In this question, denote by K the Cartesian coordinate system with vector basis $\mathbf{i}_1,\ \mathbf{i}_2,\ \mathbf{i}_3$, the standard orthonormal basis of $\mathbb{R}^3$. Denote by K' the coordinate system with vector basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ given by

$$\begin{aligned}\mathbf{e}_1 &= \mathbf{i}_1\\ \mathbf{e}_2 &= \mathbf{i}_1 - \mathbf{i}_3\\ \mathbf{e}_1 &= \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3.\end{aligned}$$

(a) Find the dual basis $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$. [8 marks]

(b) Find the covariant and contravariant components of the vector $\mathbf{V} = 2\mathbf{i}_1+\mathbf{i}_2+2\mathbf{i}_3$ with respect to the bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$. [8 marks]

(c) Consider the second-order tensor of K with components [9 marks]

$$[P_{ik}] = [P^{ik}] = [P_i^{\cdot k}] = [P^i_{\cdot k}] = \begin{pmatrix} 0 & 1 & 1\\ 1 & 0 & -1\\ -1 & 1 & 1\end{pmatrix}.$$

Express the covariant components of the given tensor in the coordinate system K'.

Answers:

(a) The dual basis is

$$\mathbf{e}^1 = \begin{pmatrix} 1\\ -2\\ 1\end{pmatrix} \quad \mathbf{e}^2 = \begin{pmatrix} 0\\ 1\\ -1\end{pmatrix} \quad \mathbf{e}^3 = \begin{pmatrix} 0\\ 1\\ 0\end{pmatrix}.$$

(b) Covariant components:

$$V_1 = 2, \quad V_2 = 0, \quad V_3 = 5.$$

Contravariant components:

$$V_1 = 2, \quad V_2 = -1, \quad V_3 = 1.$$

(c)

$$[P'_{jk}] = \begin{pmatrix} 0 & -1 & 2\\ 1 & 1 & 1\\ 0 & -1 & 3\end{pmatrix}.$$

Question 3

Let A_{ij} and B_{ijk} be tensors in a three dimensional generalised coordinate system.

(a) Write down the transformation rules for A_{ij} and B_{ijk}. [8 marks]

(b) Using the transformation rules for A_{ij} and B_{ijk}, prove that the inner product $A_{ij}B_{ikl}$ is a tensor and state its rank. [8 marks]

[9 marks]

(c) (i) Show that $\epsilon_{ijk}\epsilon_{ij\ell} = 2\delta_{k\ell}$.

(ii) Suppose the tensors A_{ij} and B_{ijk} are related by the rule

$$B_{ijk} = \epsilon_{ij\ell}A_{\ell k}.$$

Find an equation for $A_{\ell k}$ in terms of B_{ijk}. (Hint: Use the equality in (c)(i).)

Answers:

(a) $A'_{ij} = L^m_{i'}L^n_{j'}A_{mn}$ and $B'_{ijk} = L^m_{i'}L^n_{j'}L^p_{k'}B_{mnp}$, where $L^s_{r'}$ are the coefficients of the direct transformation.

(b) Rank three: $A_{ij}B_{ik\ell} = L^n_{j'}L^q_{k'}L^r_{\ell'}A_{mn}B_{mqr}$.

(c) $A_{mk} = \frac{1}{2}\epsilon_{ijm}B_{ijk}$.

Question 4

The parabolic cylindrical coordinates are the three-dimensional orthogonal coordinate system $(x^1, x^2, x^3) = (\sigma, \tau, \phi)$ with position $\mathbf{r}$ given by

$$\mathbf{r} = \sigma\tau\cos(\phi)\mathbf{i}_1 + \sigma\tau\sin(\phi)\mathbf{i}_2 + \tfrac{1}{2}(\sigma^2 - \tau^2)\mathbf{i}_3,$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are the usual Cartesian basis vectors.

(a) Compute the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of the parabolic coordinate system. [9 marks]

(b) Compute the metric coefficients of the arc length and the components of the covariant metric tensor g_{ii} for the parabolic coordinate system. [8 marks]

(c) Determine the following Christoffel symbols of the first kind for the parabolic coordinate system: [8 marks]

$$\Gamma_{112}, \quad \Gamma_{123}, \quad \Gamma_{122}, \quad \Gamma_{233}.$$

Answers:

(a)

$$\begin{aligned}
\mathbf{e}_1 &= s\cos(\phi)\mathbf{i}_1 + s\sin(\phi)\mathbf{i}_2 + r\mathbf{i}_3 \\
\mathbf{e}_2 &= r\cos(\phi)\mathbf{i}_1 + r\sin(\phi)\mathbf{i}_2 - s\mathbf{i}_3 \\
\mathbf{e}_3 &= -rs\sin(\phi)\mathbf{i}_1 + rs\cos(\phi)\mathbf{i}_2.
\end{aligned}$$

(b) Arc length: $(ds)^2 = (\sigma^2 + \tau^2)(d\sigma)^2 + (\sigma^2 + \tau^2)(d\tau)^2 + r^2s^2(d\phi)^2$.

Metric coefficients: $h_1 = \sqrt{\sigma^2 + \tau^2}$, $h_2 = \sqrt{\sigma^2 + \tau^2}$, $h_3 = \sigma\tau$.

Nonzero components of the metric tensor: $g_{11} = \sigma^2 + \tau^2$, $g_{22} = \sigma^2 + \tau^2$, $g_{33} = \sigma^2\tau^2$.

(c)

$$\begin{aligned}
\Gamma_{112} &= \tau, \\
\Gamma_{123} &= 0, \\
\Gamma_{122} &= -\sigma, \\
\Gamma_{233} &= -\sigma^2\tau.
\end{aligned}$$
