UNIVERSITY OF
LINCOLN
UNITED KINGDOM

In-Class Test: Final Examination

School	**Mathematics and Physics**
Module Title	**Tensor Analysis**
Module Code	**MTH3008**
Module Coordinator	**Paula M Lins de Araujo**
Completion time	**2 hours**
Scanning and uploading time	**30 minutes**
Academic term	**End of 2nd term 2022/2023**

General Instructions to Candidates

1. In sitting this examination you agree to **comply** with the University of Lincoln Code of Conduct in Examinations.
2. This assessment is a **closed book exam**, but you are allowed to bring **one piece of A4 paper** with **hand-written** notes on both sides.
3. You must submit only one version of your solution. You may cross out all workings that you do not want to be marked.
4. The examination **must** be completed within the **completion time** indicated above. Continuing past this time will be classified as misconduct in examinations.
5. The **completion time** may be different for students with **Personal Academic Study Support (PASS)**.
6. Extensions do not apply, but Mitigating Circumstances can be applied for in the normal way.
7. After completion you must **scan** your work clearly (without blur) and in the correct upright portrait orientation for each page.
8. You **must** submit your answers as a PDF to Turnitin on Blackboard **within 30 minutes of completion**:
9. You must **return** the exam script and your written answers to an invigilator.
10. **No collaboration or interaction** with other individuals using any means of communication or device is permitted during the examination.
11. All work will be **subject to plagiarism and academic integrity checks**. In submitting your assessment you are claiming that it is your own original work; if standard checks suggest otherwise, Academic Misconduct Regulations will be applied.

Module-specific instructions *(to be edited according to each module's needs)*

1. You must **show all your workings**
2. **Answer all questions** (or the usual sentence in case of question types A and B)

Question 1

(a) Compute the outer product $M \otimes N$ of the following two matrices: [9 marks]

$$M = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \qquad \text{and } N = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{pmatrix}.$$

(b) Suppose $T_{ijk\ell}$ is a rank-four tensor. Using the transformation rule, show that T_{ijij} is a scalar. [8 marks]

(c) Using suffix notation, find an expression involving no cross products for [8 marks]

$$(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{z}).$$

Question 2

In this question, denote by K the Cartesian coordinate system with vector basis $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, the standard orthonormal basis of $\mathbb{R}^3$. Denote by K' the coordinate system with vector basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ given by

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{i}_1 \\ \mathbf{e}_2 &= \mathbf{i}_1 - \mathbf{i}_3 \\ \mathbf{e}_1 &= \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3. \end{aligned}$$

(a) Find the dual basis $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$. [8 marks]

(b) Find the covariant and contravariant components of the vector $\mathbf{V} = 2\mathbf{i}_1 + \mathbf{i}_2 + 2\mathbf{i}_3$ with respect to the bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$. [8 marks]

(c) Consider the second-order tensor of K with components [9 marks]

$$[P_{ik}] = [P^{ik}] = [P_i^{\cdot k}] = [P^i_{\cdot k}] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Express the covariant components of the given tensor in the coordinate system K'.

Question 3

Let A_{ij} and B_{ijk} be tensors in a three dimensional generalised coordinate system.

(a) Write down the transformation rules for A_{ij} and B_{ijk}. [8 marks]

(b) Using the transformation rules for A_{ij} and B_{ijk}, prove that the inner product $A_{ij}B_{ikl}$ is a tensor and state its rank. [8 marks]

[9 marks]

(c) (i) Show that $\epsilon_{ijk}\epsilon_{ij\ell} = 2\delta_{k\ell}$.

(ii) Suppose the tensors A_{ij} and B_{ijk} are related by the rule

$$B_{ijk} = \epsilon_{ij\ell}A_{\ell k}.$$

Find an equation for $A_{\ell k}$ in terms of B_{ijk}. (Hint: Use the equality in (c)(i).)

Question 4

The parabolic cylindrical coordinates are the three-dimensional orthogonal coordinate system $(x^1, x^2, x^3) = (\sigma, \tau, \phi)$ with position $\mathbf{r}$ given by

$$\mathbf{r} = \sigma\tau\cos(\phi)\mathbf{i}_1 + \sigma\tau\sin(\phi)\mathbf{i}_2 + \tfrac{1}{2}(\sigma^2 - \tau^2)\mathbf{i}_3,$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are the usual Cartesian basis vectors.

(a) Compute the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of the parabolic coordinate system. [9 marks]

(b) Compute the metric coefficients of the arc length and the components of the covariant metric tensor g_{ii} for the parabolic coordinate system. [8 marks]

(c) Determine the following Christoffel symbols of the first kind for the parabolic coordinate system: [8 marks]

$$\Gamma_{112}, \quad \Gamma_{123}, \quad \Gamma_{122}, \quad \Gamma_{233}.$$