

## Question 1

- (a) Compute the outer product  $M \otimes N$  of the following two matrices: [9 marks]

$$M = \begin{pmatrix} 0 & 1 \\ 7 & 2 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

- (b) Suppose  $T_{ijkl}$  is a rank-four tensor. [8 marks]

- (i) Write down the transformation rule of  $T_{ijkl}$ .  
(ii) Using the transformation rule, show that  $T_{ijij}$  is a scalar.

- (c) Let  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$ , and  $\mathbf{z}$  be vectors in  $\mathbb{R}^3$ . Using suffix notation, find an expression [8 marks]  
involving no cross products for

$$(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) \cdot \mathbf{z}.$$

Write your final answer in **vector notation**. Provide all steps of your workings.

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**Answers: (a)**

$$M \otimes N = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 7 & 0 & 2 & 0 \\ 7 & -7 & 2 & -2 \\ -7 & 14 & -2 & 4 \end{pmatrix}$$

**(b)(i)**  $T'_{ijkl} = L_{im}L_{jn}L_{kp}L_{lq}T_{mnpq}$ .

**(b)(ii)** No numerical solution.

**(c)**  $(\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{z}) - (\mathbf{u} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{z})$ .

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## Question 2

In this question, denote by  $K$  the Cartesian coordinate system with vector basis  $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ , the standard orthonormal basis of  $\mathbb{R}^3$ . Denote by  $K'$  the coordinate system with vector basis  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  given by

$$\mathbf{e}_1 = 2\mathbf{i}_1 + \mathbf{i}_3$$

$$\mathbf{e}_2 = \mathbf{i}_2$$

$$\mathbf{e}_3 = \mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3.$$

(a) Find the dual basis  $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ . [8 marks]

(b) Find the covariant and contravariant components of the vector  $\mathbf{V} = 3\mathbf{i}_1 - 2\mathbf{i}_2 + \mathbf{i}_3$  with respect to the bases  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  and  $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ . [8 marks]

(c) Consider the second-order tensor of  $K$  with components [9 marks]

$$[P_{ik}] = [P^{ik}] = [P_i{}^k] = [P^i{}_k] = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Express the covariant components of the given tensor in the coordinate system  $K'$ .

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### Answers:

(a)

$$\mathbf{e}^1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\mathbf{e}^2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$$\mathbf{e}^3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

(b)

$$V_1 = 7, \quad V_2 = -2, \quad V_3 = 6.$$

$$V^1 = 2, \quad V^2 = -3, \quad V^3 = -1.$$

(c)

$$\begin{pmatrix} -1 & 4 & -4 \\ 5 & 0 & 2 \\ -6 & 2 & -4 \end{pmatrix}.$$

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### Question 3

Consider the three-dimensional orthogonal coordinate system  $(x^1, x^2, x^3) = (\sigma, \tau, \phi)$  with position  $\mathbf{r}$  given by

$$\mathbf{r} = \sigma\tau\mathbf{i}_1 + \frac{1}{2}(\sigma^2 - \tau^2)\mathbf{i}_2 + \phi\mathbf{i}_3,$$

where  $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$  are the usual Cartesian basis vectors.

- (a) Compute the basis vectors  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  of this coordinate system. [9 marks]
- (b) Compute the metric coefficients of the arc length and the components of the covariant metric tensor  $g_{ii}$  for the parabolic coordinate system. [8 marks]
- (c) Determine the following Christoffel symbols of the first kind for the parabolic coordinate system: [8 marks]

$$\Gamma_{112}, \Gamma_{123}, \Gamma_{122}, \Gamma_{233}.$$

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#### Answers:

(a)

$$\mathbf{e}_1 = \tau\mathbf{i}_1 + \sigma\mathbf{i}_2$$

$$\mathbf{e}_2 = \sigma\mathbf{i}_1 - \tau\mathbf{i}_2$$

$$\mathbf{e}_3 = \mathbf{i}_3.$$

(b)

$$(ds)^2 = (\sigma^2 + \tau^2)(d\sigma)^2 + (\sigma^2 + \tau^2)(d\tau)^2 + (d\phi)^2.$$

and

$$h_1 = \sqrt{\sigma^2 + \tau^2}, \quad h_2 = \sqrt{\sigma^2 + \tau^2}, \quad h_3 = 1.$$

(c)

$$\Gamma_{112} = \tau,$$

$$\Gamma_{123} = 0,$$

$$\Gamma_{122} = -\sigma,$$

$$\Gamma_{233} = 0.$$

### Question 4

(a) The third-rank tensor  $T_{ijk}$  is symmetric with respect to its last two suffixes but antisymmetric with respect to its first and second suffices. Show that all entries of this tensor are zero (i.e.  $T_{ijk} = 0$ , for all choices of  $i, j, k$ ). [8 marks]

(b) Let  $V_i$  be a (non-constant) covariant tensor. That is, it transforms according to the rule  $V'_i = L^{k}_{i'} V_k$ . Prove that  $\frac{\partial V_i}{\partial x^j}$  is not a tensor, i.e. that it does not satisfy the transformation law of a tensor. You may use the formulas [8 marks]

$$L^{k}_{i'} = \frac{\partial x^k}{\partial x'^{i'}}, \quad L^{i'}_k = \frac{\partial x'^{i'}}{\partial x^k}.$$

(c) Let  $f$  be a scalar field. Using suffix notation, evaluate the following expression [9 marks]

$$\nabla \times (\nabla f).$$

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#### Answers:

(a) No numerical solution.

(b) No numerical solution.

(c) No numerical solution.

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