UNIVERSITY OF
LINCOLN
UNITED KINGDOM

In-Class Test: Final Examination

School	**Mathematics and Physics**
Module Title	**Tensor Analysis**
Module Code	**MTH3008**
Module Coordinator	**Paula M Lins de Araujo**
Completion time	**2 hours**
Scanning and uploading time	**30 minutes**
Academic term	**End of 2nd term 2023/2024**

General Instructions to Candidates

1. In sitting this examination you agree to **comply** with the University of Lincoln Code of Conduct in Examinations.
2. This assessment is a **closed book exam**, but you are allowed to bring **one piece of A4 paper** with **hand-written** notes on both sides.
3. You must submit only one version of your solution. You may cross out all workings that you do not want to be marked.
4. The examination **must** be completed within the **completion time** indicated above. Continuing past this time will be classified as misconduct in examinations.
5. The **completion time** may be different for students with **Personal Academic Study Support (PASS)**.
6. Extensions do not apply, but Mitigating Circumstances can be applied for in the normal way.
7. After completion you must **scan** your work clearly (without blur) and in the correct upright portrait orientation for each page.
8. You **must** submit your answers as a PDF to Blackboard **within 30 minutes of completion (**Microsoft Lens can be used to scan the papers**)**:
9. You must **return** the exam script and your written answers to an invigilator.
10. **No collaboration or interaction** with other individuals using any means of communication or device is permitted during the examination.
11. All work will be **subject to plagiarism and academic integrity checks**. In submitting your assessment you are claiming that it is your own original work; if standard checks suggest otherwise, Academic Misconduct Regulations will be applied.

Module-specific instructions

1. You must **show all your workings.**
2. **Answer all questions.**
3. The use of any electronic devices (including calculators) is not allowed.

Question 1

(a) Compute the outer product $M \otimes N$ of the following two matrices: [9 marks]

$$M = \begin{pmatrix} 0 & 1 \\ 7 & 2 \end{pmatrix} \quad \text{and } N = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

(b) Suppose $T_{ijk\ell}$ is a rank-four tensor. [8 marks]

(i) Write down the transformation rule of $T_{ijk\ell}$.

(ii) Using the transformation rule, show that T_{ijij} is a scalar.

(c) Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{z}$ be vectors in $\mathbb{R}^3$. Using suffix notation, find an expression involving no cross products for [8 marks]

$$(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})) \cdot \mathbf{z}.$$

Write your final answer in **vector notation**. Provide all steps of your workings.

Question 2

In this question, denote by K the Cartesian coordinate system with vector basis $\mathbf{i}_1,\ \mathbf{i}_2,\ \mathbf{i}_3$, the standard orthonormal basis of $\mathbb{R}^3$. Denote by K' the coordinate system with vector basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ given by

$$\begin{aligned}\mathbf{e}_1 &= 2\mathbf{i}_1 + \mathbf{i}_3\\ \mathbf{e}_2 &= \mathbf{i}_2\\ \mathbf{e}_3 &= \mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3.\end{aligned}$$

(a) Find the dual basis $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$. [8 marks]

(b) Find the covariant and contravariant components of the vector $\mathbf{V} = 3\mathbf{i}_1 - 2\mathbf{i}_2 + \mathbf{i}_3$ with respect to the bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$. [8 marks]

(c) Consider the second-order tensor of K with components [9 marks]

$$[P_{ik}] = [P^{ik}] = [P_{i}^{\cdot k}] = [P_{\cdot k}^{i}] = \begin{pmatrix} 0 & 2 & 0 \\ 3 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}.$$

Express the covariant components of the given tensor in the coordinate system K'.

Question 3

Consider the three-dimensional orthogonal coordinate system $(x^1, x^2, x^3) = (\sigma, \tau, \phi)$ with position $\mathbf{r}$ given by

$$\mathbf{r} = \sigma\tau\mathbf{i}_1 + \tfrac{1}{2}(\sigma^2 - \tau^2)\mathbf{i}_2 + \phi\mathbf{i}_3,$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are the usual Cartesian basis vectors.

(a) Compute the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of this coordinate system. [9 marks]

(b) Compute the metric coefficients of the arc length and the components of the covariant metric tensor g_{ii} for the parabolic coordinate system. [8 marks]

(c) Determine the following Christoffel symbols of the first kind for the parabolic coordinate system: [8 marks]

$$\Gamma_{112}, \quad \Gamma_{123}, \quad \Gamma_{122}, \quad \Gamma_{233}.$$

Question 4

(a) The third-rank tensor T_{ijk} is symmetric with respect to its last two suffixes but antisymmetric with respect to its first and second suffices. Show that all entries of this tensor are zero (i.e. $T_{ijk} = 0$, for all choices of i,j,k). [8 marks]

(b) Let V_i be a (non-constant) covariant tensor. That is, it transforms according to the rule $V_i' = L_{i'}^k V_k$. Prove that $\frac{\partial V_i}{\partial x^j}$ is not a tensor, i.e. that it does not satisfy the transformation law of a tensor. You may use the formulas [8 marks]

$$L_{i'}^k = \frac{\partial x^k}{\partial x'^i}, \quad L_k^{i'} = \frac{\partial x'^i}{\partial x^k}.$$

(c) Let f be a scalar field. Using suffix notation, evaluate the following expression [9 marks]

$$\nabla \times (\nabla f).$$