2024-2025 MTH3008 Exam Final Answers

Question 1

[25 marks]

(a) In this item, we consider the tensors [8 marks]

$$[T_{ik}] = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 1 & -1 & 4 \end{pmatrix}, \qquad \mathbf{A} = \mathbf{i}_1 - \mathbf{i}_2 + \mathbf{i}_3, \qquad \mathbf{B} = \mathbf{i}_1 - 2\mathbf{i}_2.$$

Find the inner products

(i) $T_{ik}A_k$.

(ii) $T_{ik}A_kB_i$.

(b) Let T_{ijk} and R_{ijk} be tensors of rank 3. Using the **transformation rule** (i.e., **formal definition**) of a tensor, show that [9 marks]

$$T_{ipj}R_{ijq}$$

is a rank 2 tensor.

Solutions based on number of free indices will not be accepted. Use the formal definition, i.e. the transformation rule.

(c) Let A be a 3×3 matrix over $\mathbb{R}$. Use **suffix notation** to show that if A has two identical rows, then its determinant $|A|$ is zero. [8 marks]

Here, you must demonstrate this fact using suffix notation. General linear algebra arguments will not be accepted

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Answers:

(ai) $\begin{pmatrix} 3 \\ -1 \\ 6 \end{pmatrix}$.

(aii) 5

(b) No numerical solution.

(c) No numerical solution.

Question 2

[25 marks]

In this question, denote by K the Cartesian coordinate system with vector basis $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, the standard orthonormal basis of $\mathbb{R}^3$. Denote by K' the coordinate system with vector basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ given by

$$\mathbf{e}_1 = 3\mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$$
$$\mathbf{e}_2 = \mathbf{i}_2$$
$$\mathbf{e}_3 = 2\mathbf{i}_1 + \mathbf{i}_3.$$

(a) Find the basis $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ dual to $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. [9 marks]

(b) (i) Find the covariant components of the vector $\mathbf{V} = \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3$ with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. [3 marks]

(ii) Find the contravariant components of the vector $\mathbf{V} = \mathbf{i}_1 + \mathbf{i}_2 - 2\mathbf{i}_3$ with respect to the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. [3 marks]

(c) (i) Find the components of the rotation matrix $L = (L^j_{i'})$. [4 marks]

(ii) Express the covariant components of the second-order tensor of K with components [6 marks]

$$[P_{ik}] = [P^{ik}] = [P_i^{\cdot k}] = [P^i_{\cdot k}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

in the coordinate system K'.

Do not forget to show your workings and justify your answers.

Answers:

(a)

$$\mathbf{e}^1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \qquad \mathbf{e}^2 = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}, \qquad \mathbf{e}^3 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}.$$

(bi)

$$V_1 = 2, \qquad V_2 = 1, \qquad V_3 = 0.$$

(bii)

$$V^1 = 5, \qquad V^2 = -4, \qquad V^3 = -7.$$

(ci) $L = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

(cii) $\begin{pmatrix} 13 & 2 & 8 \\ 1 & 2 & -1 \\ 9 & 0 & 7 \end{pmatrix}$.

Question 3

[25 marks]

(a) Let B_{ik} be a second-rank tensor with $B_{ik} \neq 0$ for all $i, k \in \{1, 2, 3\}$. We define a quantity A_{ijk} by

$$A_{ijk} = \epsilon_{ij\ell} B_{k\ell}.$$

The quantity A_{ijk} is a third-rank tensor (but you do not need to show this).

Check the symmetry property of A_{ijk} with respect to the following indices. Specifically, determine whether A_{ijk} is symmetric, antisymmetric, or neither with respect to the following indices.

(i) The first and second indices, [4 marks]

(ii) The first and the last indices. [4 marks]

(b) Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be vectors in $\mathbb{R}^3$. Using suffix notation, find an expression involving no cross products for [9 marks]

$$(\mathbf{a} \times (\mathbf{b} \times \mathbf{c})) \cdot ((\mathbf{a} \cdot \mathbf{c})\, \mathbf{d}).$$

Write your final answer in **vector notation** and provide all steps of your workings.

(c) Let $\mathbf{r} = (x_1, x_2, x_3)$ denote the position vector in a coordinate system in $\mathbb{R}^3$. Define $\underline{r} = |\mathbf{r}|$ and let f be a scalar field. Recall that ∇ denotes the gradient operator. [8 marks]

Show that

$$\nabla f(\underline{r}) = f'(\underline{r}) \frac{\mathbf{r}}{\underline{r}}.$$

[Hint: Recall that $f'(\underline{r}) = \frac{\partial f}{\partial \underline{r}}$.]

Answers:

(ai) A_{ijk} is antisymmetric with respect to the first two indices.

(aii) A_{ijk} is neither symmetric not antisymmetric with respect to the first and last indices.

(b) $(\mathbf{a} \cdot \mathbf{c})^2 (\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{b})(\mathbf{a} \cdot \mathbf{c})(\mathbf{c} \cdot \mathbf{d})$.

(c) No numerical solution.

Question 4

[25 marks]

In this question, consider the three-dimensional orthogonal coordinate system $(x^1, x^2, x^3) = (\rho,\ \phi,\ \theta)$ with position r given by

$$\mathbf{r} = e^{\rho}\cos\phi\cos\theta\mathbf{i}_1 + e^{\rho}\sin\phi\cos\theta\mathbf{i}_2 + e^{\rho}\sin\theta\mathbf{i}_3,$$

where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are the usual Cartesian basis vectors.

(a) Compute the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of this coordinate system. [9 marks]

(b) (i) Compute the metric tensor g_{ik} of this coordinate system. [6 marks]

(ii) Describe the arc length element in terms of the metric coefficients. [2 marks]

(c) Determine the following Christoffel symbols of the first kind for the given coordinate system: [8 marks]

$$\Gamma_{111}, \qquad \Gamma_{122}, \qquad \Gamma_{223}, \qquad \Gamma_{233}.$$

Answers:

(a)

$$\begin{aligned}
\mathbf{e}_1 &= e^{\rho}\cos\phi\cos\theta\mathbf{i}_1 + e^{\rho}\sin\phi\cos\theta\mathbf{i}_2 + e^{\rho}\sin\theta\mathbf{i}_3,\\
\mathbf{e}_2 &= -e^{\rho}\sin\phi\cos\theta\mathbf{i}_1 + e^{\rho}\cos\phi\cos\theta\mathbf{i}_2,\\
\mathbf{e}_3 &= -e^{\rho}\cos\phi\sin\theta\mathbf{i}_1 - e^{\rho}\sin\phi\sin\theta\mathbf{i}_2 + e^{\rho}\cos\theta.
\end{aligned}$$

(bi) $g_{ij} = 0$ if $i \neq j$ and

$$g_{11} = e^{2\rho}, \qquad g_{22} = e^{2\rho}\cos^2\theta, \qquad g_{33} = e^{2\rho}, \qquad g_{ij} = 0 \text{ whenever } i \neq j.$$

(bii) $(ds)^2 = (e^{\rho}d\rho)^2 + (\rho\cos\theta d\phi)^2 + (e^{\rho}d\theta)^2$ and $h_1 = e^{\rho}, \qquad h_2 = e^{\rho}\cos\theta, \qquad h_3 = e^{\rho}.$

(c) $\Gamma_{111} = 2e^{2\rho}, \qquad \Gamma_{122} = 2e^{2\rho}\cos^2\theta, \qquad \Gamma_{223} = -2e^{2\rho}\cos\theta\sin\theta, \qquad \Gamma_{233} = 0.$
