Tensor Analysis

Slides week 19 – Lecture 1

Paula Lins

University of Lincoln

2025/26

Welcome to MTH3008: Tensor Analysis!

Lecturer

- Dr Paula Lins
- pmacedolinsdearaujo@lincoln.ac.uk

Module information

Weekly resources on Blackboard

Each week, you will find the following materials on Blackboard:

- **Lecture slides**
  - Slides will be available in advance, in the previous week.
- **Exercises**
  - Problem sheets will be made available each week.
  - You are expected to attempt **all** exercises **before the following lecture**.
  - Some exercises will be solved in class.
  - Solutions will be uploaded together with the lecture slides in the following week.
  - Please make a serious attempt to solve the exercises before looking at the solutions!

Material

- The slides will cover everything you need to know for the assessments.
- You can find further references on BB: Module content $\rightarrow$ Module resources $\rightarrow$ Module reading list.

Asking questions

- Ask questions at any time.
- When solving exercises, use the opportunity to ask questions privately.

Assessments

Assessments

- **Portfolio (40 %)**
  - Part A at home
  - Part B: In-class test
- **Final Exam (60 %)**

Remark

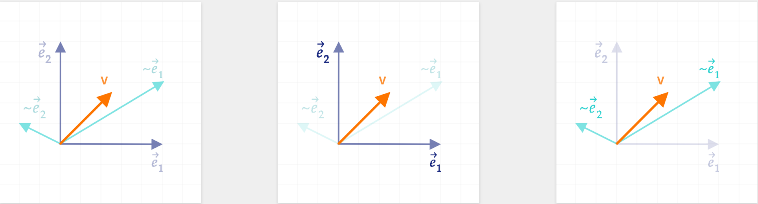
1. Part A will be submitted **together** with part B as one assessement.
2. For in-class assessments: You are allowed to bring one piece of A4 paper with hand-written or typed notes on both sides.

Tensors

Tensors

"Tensors are mathematical objects that are invariant under a change of coordinates & have components that change in predictable ways."

Jesus Najera



Originally Published:
https://www.setzeus.com/public-blog-post/a-light-intro-to-tensors

What is a tensor?

A tensor is simply a **generalisation of a vector** . . .

Why are they important?

Tensor are a mathematical tool that allows us to express physical quantities e.g.

- temperature - scalar (rank 0 tensor)
- velocity - vector (rank 1 tensor)
- stress - matrix (rank 2 tensor)

independently of the coordinate system!

Motivation: Einstein's General Relatively equation is written exclusively in terms of tensors.

Syllabus

Blackboard: Module Content → Module Resources → Syllabus

- Chapter 1: Suffix Notation
- Chapter 2: Vector Differential Operators
- Chapter 3: Local Coordinate Transfrom
- Chapter 4: Tensors
- Chapter 5: Tensors in a generalised coordinate system
- Chapter 6: Tensor Algebra
- Chapter 7: Tensor Fields

Let us start!

I hope you enjoy this module!!

Chapter 1: Suffix Notation

Today: Chapter 1–Suffix Notation

1. Suffix Notation
2. The Kronecker Delta
3. The Alternating Tensor
4. The relationship between δ_{ij} and ϵ_{ijk}

Suffix Notation

Convention

Our convention

Here, (unless stated otherwise) we assume that all vectors are **real** and **three-dimensional**.

This means a **vector** is triple of real numbers

$$\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3.$$

We can also write

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$ are unit vectors.

We use **suffix notation** to ease notation.

For instance, consider the dot product of two vectors:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{j=1}^{3} a_j b_j.$$

In suffix (or index) notation:

$$\mathbf{a} \cdot \mathbf{b} = a_j b_j, \quad \text{(i.e. \textbf{drop the sum.})}$$

- j is a repeated **dummy index**;
- Repeated indices are always summed over for $j = 1, 2, 3$;
- Dummy indices must NOT appear more than twice in any term;
- Choice of index does not matter

$$a_j b_j = a_k b_k = a_n b_n = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Example

Another example where we can see how **suffix notation** can ease notation is a multiplication of dot products.

Given vectors $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$, $\mathbf{c} = (c_1, c_2, c_3)$ and $\mathbf{d} = (d_1, d_2, d_3)$, we have

$$\begin{aligned}(\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) &= (a_1 b_1 + a_2 b_2 + a_3 b_3)(c_1 d_1 + c_2 d_2 + c_3 d_3) \\ &= \left(\sum_{j=1}^{3} a_j b_j \right) \left(\sum_{k=1}^{3} c_k d_k \right).\end{aligned}$$

In suffix notation

$$(\mathbf{a} \cdot \mathbf{b})(\mathbf{c} \cdot \mathbf{d}) = a_j b_j c_k d_k.$$

Again $$(\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) = a_j b_j c_k d_k$$

Remark.

- The choice of indices does not matter:

$$\begin{aligned} a_j b_j c_k d_k &= a_\ell b_\ell c_m d_m = a_z b_z c_x d_x \\ &= (a_1 b_1 + a_2 b_2 + a_3 b_3)(c_1 d_1 + c_2 d_2 + c_3 d_3). \end{aligned}$$

- The order doesn't matter either:

$$c_k a_j d_k b_j = (a_j b_j)(c_k d_k) \quad \text{indices indicate operations.}$$

- Make sure each index appears at most twice in each term!

Free index vs Dummy index

Free index vs Dummy index

- **Free indices:** Occur once a_i, b_{jk} or $c_{k\ell m}$
- **Dummy indices:** Repeated indices (**exactly** twice) $a_\ell b_\ell$ or $a_i b_i c_j d_j$.

Example

How many free/dummy indices are there in:

$$a_i b_j c_k d_\ell e_\ell ?$$

- 3 free indices i, j, and k,
- 1 dummy index ℓ.

Free index vs Dummy index-Examples

Free indices

Free indices represent entries of a vector. For instance, if $\mathbf{v} = (v_1, v_2, v_3)$ and $\lambda \in \mathbb{R}$, we know that

$$\lambda \mathbf{v} = (\lambda v_1, \lambda v_2, \lambda v_3).$$

In index notation we could just write

$$(\lambda \mathbf{v})_i = \lambda v_i.$$

So, v_i represents each entry of $\mathbf{v}$.

Important

When we write $(\lambda \mathbf{v})_i$ we mean the ith component of the vector $\lambda \mathbf{v}$.

Dummy indices

Dummy indices represent sums: given $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{u} = (u_1, u_2, u_3)$, the dot product is

$$\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2 + v_3 u_3,$$

In suffix notation

$$\mathbf{v} \cdot \mathbf{u} = v_i u_i.$$

Scalars

A **scalar** is an element of $\mathbb{R}$.

We can recognise a scalar in suffix notation because it has **zero** free indices.

Example.

The dot product produces a **scalar** quantity:

$$(1,2,3)\cdot(2,2,2) = 1\cdot 2 + 2\cdot 2 + 3\cdot 2 = 12.$$

We can use suffix notation to easily visualise this for any two vectors **a** and **b**:

$$\mathbf{a}\cdot\mathbf{b} = a_i b_i$$

which has no free index (only a dummy index i) so it is a scalar.

Scalars vs Vectors

Question

Is the following a scalar?

$$((\mathbf{a}\cdot\mathbf{c})\mathbf{b})_i = \left(\sum_{j=1}^{3} a_j c_j\right) b_i = a_j b_i c_j.$$

We see that this has one free term, so it is not a scalar.

This is a **vector**. E.g.

$$((1,2,1)\cdot(1,0,1))\,(0,0,1) = 2(0,0,1) = (0,0,2).$$

Important! The same **free** indices must always be used for each term in an equation e.g. $(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} = \mathbf{d}$ is equivalent to $a_j c_j b_i = d_i$.

Example: A Vector Equation

Write in suffix notation: $\mathbf{u} + (\mathbf{a}\cdot\mathbf{b})\mathbf{v} = |\mathbf{a}|^2(\mathbf{b}\cdot\mathbf{v})\mathbf{a}$

First step: Introduce **free** index i:

$$(\mathbf{u} + (\mathbf{a}\cdot\mathbf{b})\mathbf{v})_i = (\underbrace{(\mathbf{a}\cdot\mathbf{a})}_{\text{rewritten}}(\mathbf{b}\cdot\mathbf{v})\mathbf{a})_i. \qquad (|\mathbf{a}| := \sqrt{\mathbf{a}\cdot\mathbf{a}})$$

Now, we "distribute" the free index i. Note that, for instance, $\mathbf{a}\cdot\mathbf{b}$ is a scalar, not a vector, so it has no free indices. So, it does not inherit i:

$$u_i + (\mathbf{a}\cdot\mathbf{b})v_i = (\mathbf{a}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{v})a_i,$$

Note: Every term has the **same** free index.

Example: A Vector Equation- second step

Example: A Vector Equation

Write in suffix notation: $\mathbf{u} + (\mathbf{a} \cdot \mathbf{b})\mathbf{v} = |\mathbf{a}|^2(\mathbf{b} \cdot \mathbf{v})\mathbf{a}$

First step: In first step we got

$$u_i + (\mathbf{a} \cdot \mathbf{b})v_i = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{v})a_i,$$

Second step: Introduce **dummy** indices. We know that $\mathbf{a} \cdot \mathbf{b}$, $\mathbf{a} \cdot \mathbf{a}$ and $\mathbf{b} \cdot \mathbf{v}$ are sums. So, they appear with dummy indices in suffix notation:

$$u_i + a_j b_j v_i = a_k a_k b_\ell v_\ell a_i$$

Important! No dummy index appears more than **twice** in any term. We used three different dummy indices j, k, ℓ.

Reminder: Suffix notation–Practice

Your turn!

Write the following in **suffix notation**:

$$(\mathbf{a} \cdot \mathbf{b})\mathbf{u} + |\mathbf{c}|^2\mathbf{v}$$

Example: Matrices

Let A and B be $n \times n$ matrices. Show that the entries of $C = AB$ can be written as

$$C_{ij} = A_{ik} B_{kj}$$

1. Think of i and j as row/column counters i.e. C_{ij} is the element in the i-th row and j-column of matrix C.

2. So if $C = AB$, C_{ij} is found by taking the i-th row of A and the j-th column of B and multiplying term by term:

$$\begin{aligned} C_{ij} &= A_{i1}B_{1j} + A_{i2}B_{2j} + \ldots + A_{in}B_{nj} \\ &= \sum_{k=1}^{n} A_{ik}B_{kj} \\ &= A_{ik}B_{kj}. \end{aligned}$$

Example: Matrices-Part 2

Example: Matrices-Part 2

For instance, for the 3×3 matrix

$$\mathbf{C} = \mathbf{AB}$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

verify $C_{ij} = A_{ik}B_{kj}$.

Remark

The formula $C_{ij} = A_{ik}B_{kj}$ for the (i, j)-component of a product of matrices will be important for us.

Example: Trace of a Matrix

Given $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$, show that:

$$\text{Trace}(\mathbf{AB}) = \text{Trace}(\mathbf{BA})$$

Trace is the sum of elements on a diagonal.

$$\text{Tr}(\mathbf{C}) = C_{11} + C_{22} + \ldots + C_{NN} = C_{jj}.$$

So we have $$\text{Tr}(\mathbf{AB}) = \text{Tr}(A_{ik}B_{kj}) = A_{jk}B_{kj}.$$

Similarly the trace of BA is

$$\text{Tr}(\mathbf{BA}) = B_{jk}A_{kj} = \underbrace{A_{kj}B_{jk}}_{\text{re-order}} = \underbrace{A_{jk}B_{kj}}_{\text{re-label}} = \text{Tr}(\mathbf{AB}).$$

Practical Question: Transpose of a Matrix

Practical Question: Transpose of a Matrix

Let A and B be the 3×3 matrices

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

Show, using suffix notation, that

$$(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T,$$

where A^T is the transpose of A.

The Kronecker Delta

Definition–Kronecker delta

Definition.

In vector notation, the **Kronecker delta** δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Remark.

- i and j each take the values 1, 2 or 3.
- δ_{ij} will reach nine values for different i and j.
- We can think of this as the identity matrix:

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

An important property of the Kronecker delta

Property of Kronecker delta

Consider the product in **suffix notation**:

$$\delta_{ij}a_j = \sum_{j=1}^{3} \delta_{ij}a_j = \delta_{i1}a_1 + \delta_{i2}a_2 + \delta_{i3}a_3, \quad i = 1, 2, 3$$

- Repeated dummy index j is summed over.
- Free index i indicates this is a vector quantity.

Next we consider each component of the vector separately i.e. $i = 1$, $i = 2$ and $i = 3$.

An important property of the Kronecker delta–Part 2

Property of Kronecker delta

Recall:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

If $i = 1$, then in **vector notation**, we have

$$\delta_{1j} a_j = \sum_{j=1}^{3} \delta_{1j} a_j = \underbrace{\delta_{11}}_{=1} a_1 + \underbrace{\delta_{12}}_{=0} a_2 + \underbrace{\delta_{13}}_{=0} a_3 = a_1.$$

An important property of the Kronecker delta–Part 2

Property of Kronecker delta

If $i = 2$, then

$$\delta_{2j}a_j = \sum_{j=1}^{3} \delta_{2j}a_j = \cancel{\delta_{21}a_1} + \delta_{22}a_2 + \cancel{\delta_{23}}a_3 = a_2$$

If $i = 3$, then

$$\delta_{3j}a_j = \sum_{j=1}^{3} \delta_{3j}a_j = \cancel{\delta_{31}a_1} + \cancel{\delta_{32}a_2} + \delta_{33}a_3 = a_3.$$

Properties of the Kronecker Delta — Part 3

Summary of Key Property

To summarise, we have:

$$\delta_{1j}a_j = a_1, \quad \delta_{2j}a_j = a_2, \quad \delta_{3j}a_j = a_3.$$

This generalises to:

$$\delta_{ij}a_j = a_i \quad \text{(the repeated index is absorbed).}$$

For this reason, the Kronecker delta is sometimes called the **substitution tensor**, because it replaces the repeated index with the free index.

Symmetry of the Kronecker Delta

Remark

The following also holds:

$$\delta_{ji} a_i = a_j.$$

Proof. **Exercise!**

Next lecture

Next time...

- More on the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

- The alternating tensor

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i, j, k \text{ are equal}, \\ +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (2,1,3) \text{ or } (3,2,1). \end{cases}$$