Tensor Analysis

Slides week 19 – Lecture 2

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Chapter 1: Suffix Notation

Today: Chapter 1–Suffix Notation

1. Suffix Notation
2. The Kronecker Delta
3. The Alternating Tensor
4. The relationship between δ_{ij} and ϵ_{ijk}

Reminder

Our convention

Here, we assume that all vectors are three-dimensional.

This means a **vector** is triple of real numbers

$$\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3.$$

We can also write

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

where $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$ are unit vectors.

Reminder: Suffix notation–example

We have seen that we can use **suffix notation** to make notation simpler.

Example

In **vector notation**, we have

$$
\begin{aligned}
(\mathbf{a}\cdot\mathbf{b})\mathbf{c} &= \left((a_1,a_2,a_3)\cdot(b_1,b_2,b_3)\right)(c_1,c_2,c_3) \\
&= \left(\sum_{i=1}^{3} a_i b_i\right)(c_1,c_2,c_3) \\
&= \left(\left(\sum_{i=1}^{3} a_i b_i\right)c_1, \left(\sum_{i=1}^{3} a_i b_i\right)c_2, \left(\sum_{i=1}^{3} a_i b_i\right)c_3\right).
\end{aligned}
$$

In **suffix notation**, we write simply

$$((\mathbf{a}\cdot\mathbf{b})\mathbf{c})_i = a_j b_j c_i.$$

Reminder: Suffix notation–steps

Recall

To write $(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$ in **suffix notation**, we follow the next steps:

First step: Introduce **free** index i:

$$((\mathbf{a}\cdot\mathbf{b})\mathbf{c})_i = (\mathbf{a}\cdot\mathbf{b})c_i$$

Second step: introduce a **dummy** index j:

$$((\mathbf{a}\cdot\mathbf{b})\mathbf{c})_i = a_j b_j c_i.$$

Reminder: Suffix notation–nomenclature

Recall

In suffix notation we have

$$((\mathbf{a} \cdot \mathbf{b})\mathbf{c})_i = a_j b_j c_i$$

- j is a repeated **dummy index**.
- Repeated indices are implicitly summed over $j = 1, 2, 3$.
- i is a **free index**.
- One free index in a term indicates it is a vector quantity.
- No free indices in a term indicate a scalar quantity.

  E.g. dot product $\mathbf{a} \cdot \mathbf{b} = a_j b_j$ produces a scalar.

Reminder: Suffix notation–rules

Recall

In suffix notation we have

$$((\mathbf{a} \cdot \mathbf{b})\mathbf{c})_i = a_j b_j c_i$$

Rules:

- No dummy index can appear more than twice in a term.
- Each term in an equation must have the same free index.

Example

Let us write the following in **suffix notation**:

$$\mathbf{u} + (\mathbf{a} \cdot \mathbf{b})\mathbf{v} = |\mathbf{a}|^2 (\mathbf{b} \cdot \mathbf{v})\mathbf{a}$$

First step: Introduce **free** index i:

$$(\mathbf{u} + (\mathbf{a} \cdot \mathbf{b})\mathbf{v})_i = (\underbrace{(\mathbf{a} \cdot \mathbf{a})}_{\text{rewritten}} (\mathbf{b} \cdot \mathbf{v})\mathbf{a})_i.$$

Then, the vectors inherit these free indices (but not the sums!):

$$u_i + (\mathbf{a} \cdot \mathbf{b}) v_i = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{v}) a_i$$

Second step: introduce **dummy indices** for sums:

$$u_i + a_j b_j v_i = a_\ell a_\ell b_m v_m a_i.$$

Example: Matrices

Let $A = (A_{ij})$ and $B = (B_{ij})$ be $n \times n$ matrices.

Then the entries of the marix $C = AB$ are given by the formula in **suffix notation**

$$C_{ij} = A_{ik}B_{kj}$$

Definition–Kronecker delta

Definition.

The **Kronecker delta** δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Remark.

- i and j each take the values 1, 2 or 3.
- δ_{ij} will reach nine values for different i and j.
- We can think of this as the identity matrix:

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Property of Kronecker delta

We have shown that

$$\delta_{ij} a_j = a_i,$$
$$\delta_{ji} a_j = a_i.$$

- That is, the repeated index is absorbed.
- For this reason, Kronecker delta is sometimes called the "**substitution tensor**", because it replaces repeated index with free index.

More on the Kronecker Delta

Kronecker Delta and the Dot Product

Using that $\delta_{ij}a_j = a_i$, we can write

$$\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a_i b_j.$$

In fact, using **suffix notation we have**

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i = a_i(\delta_{ij} b_j) = \delta_{ij} a_i b_j.$$

We can also check this using **vector notation**:

$$\delta_{ij} a_i b_j = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} a_i b_j = \delta_{11} a_1 b_1 + \delta_{22} a_2 b_2 + \delta_{33} a_3 b_3$$
$$= a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b}.$$

Vector vs suffix notation

We defined the Kronecker delta as follows

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Note that this notation is given in **vector notation**. That is, we are specifying the values of δ_{11}, δ_{12}, $\cdots$, δ_{33} at the same time.

However, in **suffix notation**, the term

$$\delta_{ii}$$

is a **sum**. (We will explore δ_{ii} in the next slide.)

So, we need to be attentive to the context.

Example 1

Example

Evaluate δ_{jj}, which is given in suffix notation.

As j is repeated, summation convention implies that we are summing from $j = 1$ to 3.

In fact, in **vector notation**, this is the same as

$$\sum_{j=1}^{3} \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3,$$

because, by definition

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Example 2

Example

Simplify the following expression in **suffix notation**

$$\delta_{ij}\delta_{jk}.$$

Solution: Here only j is repeated, so we only need to sum over j. Thus $\delta_{ij}\delta_{jk}$ in **vector notation** is

$$\sum_{j=1}^{3} \delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k}.$$

The result depend on the values of i and k.

- In fact, δ_{i1} is zero if $i \neq 1$, and 1 if $i = 1$.
- So, we know that the non-zero term among δ_{i1}, δ_{i2}, δ_{i3} is δ_{ii}.

Example

Simplify the following expression in **suffix notation**

$$\delta_{ij}\delta_{jk}.$$

Solution (cont):

- Thus

$$\delta_{ij}\delta_{jk} = \sum_{j=1}^{3} \delta_{ij}\delta_{jk} = \delta_{ii}\delta_{ik} = \delta_{ik}.$$

- We conclude

$$\boxed{\delta_{ij}\delta_{jk} = \delta_{ik}.}$$

That is, the repeated index is absorbed

Practice

Your turn!

Evaluate

$$\delta_{\ell\ell}\delta_{mn}\delta_{pp}\delta_{nq}.$$

Practical Question

Simplify the suffix notation expression

$$\delta_{ij}a_j b_\ell c_k \delta_{i\ell}$$

and write the result in **vector form**.

The alternating tensor

Next

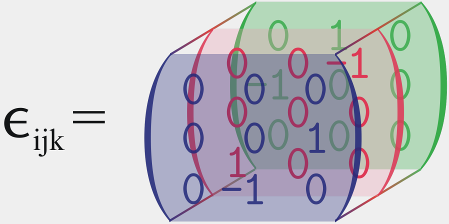
- We have seen that the Kronecker Delta can be used to define the dot product.
- We will see that the **Alternating Tensor** is useful for defining the **cross product**.

Definition.

The **alternating tensor** ϵ_{ijk} is defined by

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i, j, k \text{ are equal,} \\ +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (2,1,3) \text{ or } (3,2,1). \end{cases}$$

The **alternating tensor** ϵ_{ijk} may be visualised by a 3×3 array:



Vector vs suffix notation

- As for δ_{ij}, it is very important to distinguish **suffix** and **vector notation**.

- In the definition

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i, j, k \text{ are equal,} \\ +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (2,1,3) \text{ or } (3,2,1). \end{cases}$$

  we are using **vector notation**. So, for instance, $\epsilon_{iij} = 0$.

- In **suffix notation**, ϵ_{iij} is the sum

$$\epsilon_{iij} = \sum_{i=1}^{3} \epsilon_{iij} = \epsilon_{11j} + \epsilon_{22j} + \epsilon_{33j} = 0.$$

Properties of the Alternating Tensor

In the following, we are using **vector notation**:

- ϵ_{ijk} keeps **unchanged** if indices are reordered by a **cyclic permutation**:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}.$$

- The sign of ϵ_{ijk} **changes** if any two of the suffices are **interchanged**:

$$\epsilon_{ijk} = -\epsilon_{jik}$$

  that is, the alternating tensor ϵ_{ijk} is **anti-symmetric**.

Example- Alternating tensor

Example

Evaluate $\epsilon_{ijk}\epsilon_{ijk}$ in **suffix notation**.

- All three indices i, j and k are repeated i.e. summed over. So, in **vector notation** $\epsilon_{ijk}\epsilon_{ijk}$ is

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\epsilon_{ijk} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}^2.$$

- This gives a total of 27 terms.
- However, only six of these terms are non-zero!

$$\begin{aligned}\epsilon_{ijk}\epsilon_{ijk} &= \epsilon_{123}^2 + \epsilon_{132}^2 + \epsilon_{213}^2 + \epsilon_{231}^2 + \epsilon_{312}^2 + \epsilon_{321}^2,\\ &= 1^2 + (-1)^2 + (-1)^2 + 1^2 + 1^2 + (-1)^2 = 6.\end{aligned}$$

Relations of alternating tensors

In the following, we will see how the alternating tensor ϵ_{ijk} is related to

- The cross product of two vectors.
- The determinant of 3×3 matrices.
- The scalar triple product.

Cross Product

Definition.

Recall that, the **cross product** of two vectors

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

is

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

Cross Product

You can think of the **cross product** as follows:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}.$$

The Alternating Tensor and the Cross Product

The Alternating Tensor and the Cross Product

We can write

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k.$$

Remark.

- When we write $(\mathbf{a} \times \mathbf{b})_i$, we mean the ith component of the vector $\mathbf{a} \times \mathbf{b}$.
- Here, j and k are **dummy indices** i.e. they indicate sums.
- i is a **free index**, indicating the coordinate we are considering.

Let us check

Let us check the formula

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k$$

for $i = 1$. In **vector notation**, $\epsilon_{1jk} a_j b_k$ is the same as

$$\begin{aligned} \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{1jk} a_j b_k &= \epsilon_{123} a_2 b_3 \quad + \epsilon_{132} a_3 b_2, \\ &= (+1) a_2 b_3 + (-1) a_3 b_2, \\ &= a_2 b_3 - a_3 b_2, \end{aligned}$$

which agrees with

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k}.$$

Exercise: Check this formula for $i = 2$ and $i = 3$.

Your turn!

Use suffix notation to show that

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}.$$

Alternating Tensor and Matrix Determinant

Alternating Tensor and Matrix Determinant

There is a link between ϵ_{ijk} and the **determinant** of a 3×3 matrix:

$$|M| = \epsilon_{ijk} M_{1i} M_{2j} M_{3k}, \qquad \text{(row)}$$

$$|M| = \epsilon_{ijk} M_{i1} M_{j2} M_{k3}. \qquad \text{(column)}$$

Let us check!

Consider the matrix

$$M = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}.$$

Alternating Tensor and Matrix Determinant - Proof

Let us check!

The determinant of M is given by

$$|M| = M_{11}(M_{22}M_{33} - M_{23}M_{32}) - M_{12}(M_{21}M_{33} - M_{23}M_{31}) + M_{13}(M_{21}M_{32} - M_{22}M_{31})$$

We have

$$\epsilon_{ijk}M_{1i}M_{2j}M_{3k} = \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}M_{1i}M_{2j}M_{3k}$$
$$= \underline{\epsilon_{123}}M_{11}M_{22}M_{33} + \underline{\epsilon_{132}}M_{11}M_{23}M_{32}$$
$$+ \underline{\epsilon_{231}}M_{12}M_{23}M_{31} + \underline{\epsilon_{213}}M_{12}M_{21}M_{33}$$
$$+ \underline{\epsilon_{312}}M_{13}M_{21}M_{32} + \underline{\epsilon_{321}}M_{13}M_{22}M_{31}$$

Alternating Tensor and Matrix Determinant - Proof - Part 2

Let us check!

The determinant of M is given by

$$|M| = M_{11}(M_{22}M_{33} - M_{23}M_{32}) - M_{12}(M_{21}M_{33} - M_{23}M_{31}) + M_{13}(M_{21}M_{32} - M_{22}M_{31})$$

We have

$$\begin{aligned}\epsilon_{ijk}M_{1i}M_{2j}M_{3k} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}M_{1i}M_{2j}M_{3k} \\ &= \underline{(+1)}M_{11}M_{22}M_{33} + \underline{(-1)}M_{11}M_{23}M_{32} \\ &+ \underline{(+1)}M_{12}M_{23}M_{31} + \underline{(-1)}M_{12}M_{21}M_{33} \\ &+ \underline{(+1)}M_{13}M_{21}M_{32} + \underline{(-1)}M_{13}M_{22}M_{31}\end{aligned}$$

Alternating Tensor and Matrix Determinant - Proof- Part 3

Let us check!

The determinant of M is given by

$$|M| = M_{11}(M_{22}M_{33} - M_{23}M_{32}) - M_{12}(M_{21}M_{33} - M_{23}M_{31}) + M_{13}(M_{21}M_{32} - M_{22}M_{31})$$

We have

$$\begin{aligned} \epsilon_{ijk}M_{1i}M_{2j}M_{3k} &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}M_{1i}M_{2j}M_{3k} \\ &= M_{11}(M_{22}M_{33} - M_{23}M_{32}) \\ &+ M_{12}(M_{23}M_{31} - M_{21}M_{33}) \\ &+ M_{13}(M_{21}M_{32} - M_{22}M_{31}) = |M|. \end{aligned}$$

Alternating Tensor and Matrix Determinant - Proof- Part 4

Let us check!

Following similar steps, we can show the other equality in terms of columns:

$$|M| = \epsilon_{ijk} M_{i1} M_{j2} M_{k3}.$$

□

A RELATED FORMULA INVOLVING THE DETERMINANT

A related formula

The link between ϵ_{ijk} and the deteminant of a 3×3 matrix is

$$|M| = \epsilon_{ijk} M_{1i} M_{2j} M_{3k}.$$

An important related formula is

$$\epsilon_{pqr} |M| = \epsilon_{ijk} M_{pi} M_{qj} M_{rk}.$$

The Alternating Product and the Scalar Triple Product

Definition.

The **scalar triple product** is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Suffix notation

Let us write the **scalar triple product** in suffix notation.

Recall that the **cross product** and the **alternating tensor** are related:

$$(\mathbf{b} \times \mathbf{c})_i = \epsilon_{ijk} b_j c_k.$$

Thus,

$$\begin{aligned}\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_i (\mathbf{b} \times \mathbf{c})_i \\ &= a_i (\epsilon_{ijk} b_j c_k) \\ &= \epsilon_{ijk} a_i b_j c_k.\end{aligned}$$

Example

We now show

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \epsilon_{ijk} a_i b_j c_k && \text{(Scalar triple product)} \\
&= \epsilon_{kij} a_i b_j c_k && (\text{using } \epsilon_{ijk} = \epsilon_{kij}) \\
&= (\epsilon_{kij} a_i b_j) c_k \\
&= (\mathbf{a} \times \mathbf{b})_k c_k \\
&= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.
\end{aligned}$$

Next time ...

Next time...

- More on the Alternating Tensor
- The relationship between δ_{ij} and ϵ_{ijk}