Tensor Analysis

Slides week 20 – Lecture 1

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Chapter 1: Suffix Notation

Today: Chapter 1–Suffix Notation

1. Suffix Notation
2. The Kronecker Delta
3. The Alternating Tensor
4. The relationship between δ_{ij} and ϵ_{ijk}

Reminder

Recall: Kronecker delta

Definition.

The **Kronecker delta** δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Properties

- **Substitution tensor:**

$$\begin{aligned} \delta_{ij} a_j &= a_i, \\ \delta_{ji} a_j &= a_i. \end{aligned}$$

- **Dot product:**

$$\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a_i b_j.$$

Recall: Alternating tensor

Definition.

The **alternating tensor** ϵ_{ijk} is defined by

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i, j, k \text{ are equal,} \\ +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (2,1,3) \text{ or } (3,2,1). \end{cases}$$

Properties

In the following, we are using **vector notation**:

- ϵ_{ijk} keeps **unchanged** if indices are reordered by a **cyclic permutation**:
$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}.$$
- The sign of ϵ_{ijk} **changes** if any two of the suffices are **interchanged**:
$$\epsilon_{ijk} = -\epsilon_{jik}.$$

Relations of alternating tensors

The alternating tensor ϵ_{ijk} is related to

- The cross product of two vectors:

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k.$$

- The determinant of 3×3 matrices:

$$|M| = \epsilon_{ijk} M_{1i} M_{2j} M_{3k}, \qquad \text{(row)}$$
$$|M| = \epsilon_{ijk} M_{i1} M_{j2} M_{k3}. \qquad \text{(column)}$$

- The scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

that we will consider in the next slides.

More on ϵ_{ijk}

The alternating tensor and the Scalar Triple Product

Definition.

The **scalar triple product** is $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$.

Suffix notation

Let us write the **scalar triple product** in suffix notation.

Recall that the **cross product** and the **alternating tensor** are related:

$$(\mathbf{b} \times \mathbf{c})_i = \epsilon_{ijk} b_j c_k.$$

Thus,

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_i (\mathbf{b} \times \mathbf{c})_i \\
&= a_i (\epsilon_{ijk} b_j c_k) \\
&= \epsilon_{ijk} a_i b_j c_k.
\end{aligned}$$

Example

Let us use the scalar triple product formula to show that

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}.$$

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) &= \epsilon_{ijk}a_i b_j c_k && \text{(Scalar triple product)}\\
&= \epsilon_{kij}a_i b_j c_k && (\text{using } \epsilon_{ijk} = \epsilon_{kij})\\
&= (\epsilon_{kij}a_i b_j)c_k\\
&= (\mathbf{a}\times\mathbf{b})_k c_k\\
&= (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}.
\end{aligned}$$

Example from last time

In the previous slide, we used suffix notation to show

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

We will now show the following using suffix notation

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}).$$

Example

Let us show

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}).$$

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= \epsilon_{ijk} a_i b_j c_k && \text{(Scalar triple product)} \\
&= \epsilon_{jki} a_i b_j c_k && \text{(using } \epsilon_{ijk} = \epsilon_{jki}\text{)} \\
&= b_j \epsilon_{jki} c_k a_i && \text{(just rearranging terms)} \\
&= b_j (\epsilon_{jki} c_k a_i) \\
&= b_j (\mathbf{c} \times \mathbf{a})_j \\
&= \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}).
\end{aligned}$$

Practical Question

Your turn!

Write the vector equation

$$\mathbf{a} \times \mathbf{b} + (\mathbf{a} \cdot \mathbf{d})\mathbf{c} = \mathbf{e}$$

in suffix notation.

The Levi-Civita Symbol

There is a more general symbol called the **Levi-Civita Symbol**:

$$\epsilon_{i_1\ i_2\ i_3\ ...i_n}.$$

It is defined using the following rules:

- If any two indices are interchanged the symbol is negated.
- If any two indices are equal the symbol equals zero.

Thus, ϵ_{ijk} is just the Levi-Civita symbol in $3D$ space.

The relationship between δ_{ij} and ϵ_{ijk}

Relating δ_{ij} and ϵ_{ijk}

One can relate deltas and epsilons as follows (suffix notation).

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}.$$

Remark

- There are **four** free indices i, j, ℓ, m.
- k is a repeated dummy index, and is summed over.
- This represents 81 equations!
  (One for each choice of the quadruple i, j, ℓ, m.)

Where does this relationship come from?

Let us show some cases

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}.$$

Consider $i = 1$ and possible values for j.

- If $j = 1$, $\epsilon_{ijk} = \epsilon_{11k} = 0$. This means that LHS is 0.

  RHS is $\delta_{1\ell}\delta_{1m} - \delta_{1m}\delta_{1\ell}$ =0, as terms cancel.

- If $j = 2$, then $\epsilon_{ijk} = \epsilon_{12k} = 0$ unless $k = 3$.

  When $k = 3$, the term $\epsilon_{k\ell m}$ is zero unless $(\ell, m) = (1, 2)$ or $(\ell, m) = (2, 1)$. Thus,

$$\epsilon_{ijk}\epsilon_{k\ell m} = \epsilon_{12k}\epsilon_{k\ell m} = \begin{cases} 1, & \text{if } (\ell, m) = (1, 2) \\ -1, & \text{if } (\ell, m) = (2, 1) \\ 0, & \text{otherwise .} \end{cases}$$

Where does this relationship come from?

Let us show some cases

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}.$$

For $i = 1$, $j = 2$, $k = 3$, we have

$$\epsilon_{ijk}\epsilon_{klm} = \epsilon_{12k}\epsilon_{k\ell m} = \begin{cases} 1, & \text{if } (\ell, m) = (1, 2) \\ -1, & \text{if } (\ell, m) = (2, 1) \\ 0, & \text{otherwise} \ . \end{cases}$$

The RHS is

$$\delta_{1\ell}\delta_{2m} - \delta_{1m}\delta_{2\ell} = \begin{cases} 1, & \text{if } (\ell, m) = (1, 2) \\ -1, & \text{if } (\ell, m) = (2, 1) \\ 0, & \text{otherwise} \ . \end{cases}$$

Example 1

Use the relationship

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}.$$

to show that

$$(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i = (\mathbf{a} \cdot \mathbf{c})b_i - (\mathbf{a} \cdot \mathbf{b})c_i.$$

In fact, we know that the cross product is given by

$$(\mathbf{v} \times \mathbf{u})_i = \epsilon_{ijk}v_j u_k.$$

Thus

$$\begin{aligned}(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i &= \epsilon_{ijk}a_j(\mathbf{b} \times \mathbf{c})_k \\ &= \epsilon_{ijk}a_j\epsilon_{k\ell m}b_\ell c_m \\ &= \epsilon_{ijk}\epsilon_{k\ell m}a_j b_\ell c_m.\end{aligned}$$

Example 1

Now, we use the relationship

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}.$$

and get

$$\begin{aligned}
(\mathbf{a} \times (\mathbf{b} \times \mathbf{c}))_i &= \epsilon_{ijk}\epsilon_{k\ell m} a_j b_\ell c_m \\
&= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) a_j b_\ell c_m \\
&= \delta_{i\ell}\delta_{jm} a_j b_\ell c_m - \delta_{im}\delta_{j\ell} a_j b_\ell c_m \quad \text{(expand)} \\
&= a_m \delta_{i\ell} b_\ell c_m - a_\ell \delta_{im} b_\ell c_m \quad (e.g.\ \delta_{{\color{red}j}m} a_{\color{red}j} = a_m) \\
&= a_m b_i c_m - a_\ell b_\ell c_i, \quad (e.g.\ \delta_{i{\color{blue}\ell}} b_{\color{blue}\ell} = b_i) \\
&= (\mathbf{a} \cdot \mathbf{c}) b_i - (\mathbf{a} \cdot \mathbf{b}) c_i.
\end{aligned}$$

Practical Question

Your turn!

Simplify the following expressions (that are in suffix notation).

1. $\delta_{ij}\delta_{jk}\delta_{ki}$,
2. $\epsilon_{ijk}\epsilon_{k\ell m}\epsilon_{mni}$.

Suffix notation and ranks

Intuition: What is the rank of a tensor?

Key idea: The number of free indices determines the **rank** (or order) of a tensor.

- 0 free indices $\Rightarrow$ rank-0 tensor (a **scalar**);
- 1 free index $\Rightarrow$ rank-1 tensor (a **vector**);
- 2 free indices $\Rightarrow$ rank-2 tensor (a **matrix**);
- ...

Important remark

This is only an **intuition**. Later we will see that an object must satisfy additional transformation properties to genuinely qualify as a tensor. So this should not be taken as a formal definition of rank.

Vector differential operators

Definition

- A **scalar field** is a map that assigns a **real number** to every point in space.

$$\phi : \mathbb{R}^n \to \mathbb{R}$$

- A **vector field** is a map that assigns a **vector** to every point in space.

$$\psi : \mathbb{R}^n \to \mathbb{R}^n$$

Differential Operators

We will consider three differential operators:

- The gradient
- The divergence
- The curl

Each can be expressed in suffix notation to give more compact formulations and easier calculations.

To do this we re-label the Cartesian coordinate system (x, y, z) as

$$(x_1, x_2, x_3).$$

The Gradient

Definition.

The **gradient** of a scalar field is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right).$$

Remark.

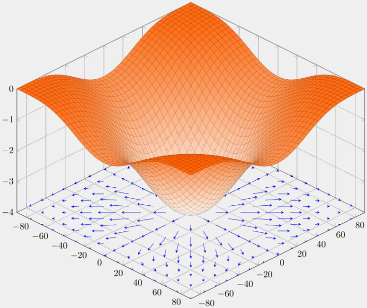
- The i-th component of the gradient is the partial derivative with respect to x_i.
- So in suffix notation we can simply write

$$[\nabla f]_i = \frac{\partial f}{\partial x_i}.$$

- The gradient of a scalar has one **free index** i, indicating the result is a **vector quantity**.

Consider first the gradient of a scalar field, ∇f:

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right).$$



E.g. the gradient of scalar field $f(x, y) = -(\cos^2 x + \cos^2 y)^2$ is shown by the vector field in the bottom plane.

Example: The Gradient

Let us find the gradient of

$$\phi(x_1, x_2, x_3) = 3x_1x_2^3 - x_2^2x_3^2$$

at the point $P = (-1, 1, 2)$.

$$\begin{aligned}\nabla\phi &= \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right), \\ &= \left(3x_2^3,\ 9x_1x_2^2 - 2x_2x_3^2,\ -2x_2^2x_3\right).\end{aligned}$$

Example: The Gradient-Part 2

Example: The Gradient

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right) = \left(3x_2^3,\, 9x_1x_2^2 - 2x_2x_3^2,\, -2x_2^2x_3\right).$$

Thus, at $P = (-1, 1, 2)$ we have

$$\begin{aligned}\nabla\phi|_{(-1,1,2)} &= \left(3x_2^3,\, 9x_1x_2^2 - 2x_2x_3^2,\, -2x_2^2x_3\right)_{(-1,1,2)} \\ &= \left(3(1)^3,\, 9(-1)(1)^2 - 2(1)(2)^2,\, -2(1)^2(2)\right), \\ &= (3, -17, -4) \\ &= 3\mathbf{i} - 17\mathbf{j} - 4\mathbf{k}.\end{aligned}$$

Note that **i**, **j**, **k** are the unit vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively.

Task: The Gradient

Your turn!

Find the gradient $\nabla\phi$ of

$$\phi(x_1, x_2, x_3) = \frac{1}{2}x_2^2x_3^3 - 3x_1x_2 + x_2^5x_3 + 1$$

at the point $P = (3, 1, 0)$.

The Divergence

Definition.

The **divergence** of a vector field $\mathbf{u}$ is

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_j}{\partial x_j} = \nabla_i u_i,$$

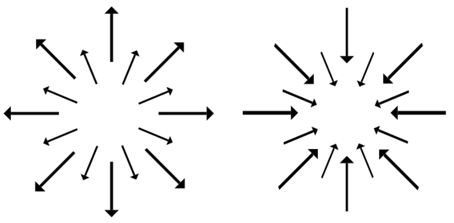
Remark.

- There is a dummy index j indicating the summation over $j = 1, 2, 3$.
- **No free indices** indicates the divergence of a vector is a **scalar quantity**.

The Divergence

The divergence of a vector field $\mathbf{u}$ is

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_j}{\partial x_j} = \nabla_i u_i,$$



E.g. $\mathrm{div}\vec{A} > 0 \rightsquigarrow$ source or $\mathrm{div}\vec{A} < 0 \rightsquigarrow$ sink

Example: The Divergence

Let us find $\nabla \cdot \mathbf{A}$ at the point $Q = (1, 1, 1)$ for

$$\mathbf{A} = x_1^2 x_3^2 \mathbf{i} - 2x_2^2 x_3^2 \mathbf{j} + x_1 x_2^2 x_3 \mathbf{k}.$$

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= \left(\frac{\partial}{\partial x_1}\mathbf{i} + \frac{\partial}{\partial x_2}\mathbf{j} + \frac{\partial}{\partial x_3}\mathbf{k} \right) \cdot \left(x_1^2 x_3^2 \mathbf{i} - 2x_2^2 x_3^2 \mathbf{j} + x_1 x_2^2 x_3 \mathbf{k} \right) \\
&= \frac{\partial}{\partial x_1}\left(x_1^2 x_3^2 \right) + \frac{\partial}{\partial x_2}\left(-2x_2^2 x_3^2 \right) + \frac{\partial}{\partial x_3}\left(x_1 x_2^2 x_3 \right) \\
&= 2x_1 x_3^2 - 4x_2 x_3^2 + x_1 x_2^2.
\end{aligned}$$

Example: The Divergence - Part 2

Example: The Divergence

Thus, at $Q = (1, 1, 1)$ we have

$$\begin{aligned}[\nabla \cdot \mathbf{A}]_{(1,1,1)} &= \left[2x_1x_3^2 - 4x_2x_3^2 + x_1x_2^2\right]_{(1,1,1)}, \\ &= 2(1)(1)^2 - 4(1)(1)^2 + (1)(1)^2 \\ &= -1.\end{aligned}$$

The divergence of a vector field is in fact a **scalar quantity**.

Your turn!

Find $\nabla \cdot \mathbf{A}$ at the point $Q = (2, 1, 2)$ for

$$\mathbf{A} = \frac{1}{2}x_1^3 x_2 \mathbf{i} - (4x_1 x_2^5 + 1)\mathbf{j} + x_2 x_3^3 \mathbf{k}.$$

Definition.

The curl of a vector field $\mathbf{u}$ is

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

Reminder: Cross product and alternating tensor

Because the **curl** uses the **cross product**, we can write it using the alternating tensor.

Reminder: alternating tensor

The **alternating tensor** ϵ_{ijk} is defined by

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i, j, k \text{ are equal,} \\ +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (2,1,3) \text{ or } (3,2,1). \end{cases}$$

Reminder: Alternating tensor and cross product

Recall that

$$(\mathbf{u} \times \mathbf{v})_i = \epsilon_{ijk} u_j v_k.$$

Curl and cross product

The components of the curl are then

$$[\nabla \times \mathbf{u}]_i = \epsilon_{ijk} \nabla_j u_k = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}.$$

Remark.

- The dummy indices j and k indicate a double sum.
- The free index i indicates the curl of a vector is a **vector quantity**.

Example: The Curl

Let us find the curl of

$$\mathbf{A} = \frac{1}{2}x_1^3 x_2 \mathbf{i} + (4x_1 x_2^5 + 1)\mathbf{j} + x_2 x_3^3 \mathbf{k}$$

at the point $Q = (2, 1, 2)$.

$$\begin{aligned}\nabla \times \mathbf{A} &= \left[\frac{\partial}{\partial x_2}\left(x_2 x_3^3\right) - \frac{\partial}{\partial x_3}\left(4x_1 x_2^5 + 1\right)\right]\mathbf{i} \\ &\quad - \left[\frac{\partial}{\partial x_1}\left(x_2 x_3^3\right) - \frac{\partial}{\partial x_3}\left(\frac{1}{2}x_1^3 x_2\right)\right]\mathbf{j} \\ &\quad + \left[\frac{\partial}{\partial x_1}\left(4x_1 x_2^5 + 1\right) - \frac{\partial}{\partial x_2}\left(\frac{1}{2}x_1^3 x_2\right)\right]\mathbf{k} \\ &= x_3^3\mathbf{i} + 0\mathbf{j} + \left(4x_2^5 - \frac{1}{2}x_1^3\right)\mathbf{k}.\end{aligned}$$

Example: The Curl

Thus, at $Q = (2, 1, 2)$ we have

$$\nabla \times \mathbf{A}|_{(2,1,2)} = \left[x_3^3 \mathbf{i} + 0\mathbf{j} + \left(4x_2^5 - \frac{1}{2} x_1^3 \right) \mathbf{k} \right]_{(2,1,2)}$$
$$= 8\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$
$$= (8, 0, 0).$$

Task: The Curl

Your turn!

Suppose
$$\mathbf{A} = x_1^2 x_3^2 \mathbf{i} - 2x_2^2 x_3^2 \mathbf{j} + x_1 x_2 x_3 \mathbf{k}.$$
Find $\nabla \times \mathbf{A}$ at the point $Q = (1, 1, 1)$.

Next time...

- Vector differential operators,
- Combinations of grad, div and curl.