Tensor Analysis

Slides week 20 – Lecture 2

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Chapter 2: Vector Differential Operators

Today: Chapter 2–Vector Differential Operators

- Vector differential operators
- Combinations of grad, div and curl
- Grad, div and curl applied to functions

Reminder

Recall: Kronecker delta

Definition.

The **Kronecker delta** δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Properties

- **Substitution tensor:**

$$\delta_{ij} a_j = a_i,$$
$$\delta_{ji} a_j = a_i.$$

- **Dot product:**

$$\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a_i b_j.$$

Recall: Alternating tensor

Definition.

The **alternating tensor** ϵ_{ijk} is defined by

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i, j, k \text{ are equal,} \\ +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (2,1,3) \text{ or } (3,2,1). \end{cases}$$

Relations of alternating tensors

The alternating tensor ϵ_{ijk} is related to

- The cross product of two vectors:

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k.$$

- The determinant of 3×3 matrices:

$$|M| = \epsilon_{ijk} M_{1i} M_{2j} M_{3k}, \qquad \text{(row)}$$
$$|M| = \epsilon_{ijk} M_{i1} M_{j2} M_{k3}. \qquad \text{(column)}$$

- The scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \epsilon_{ijk} a_i b_j c_k.$$

Relating δ_{ij} and ϵ_{ijk}

One can relate deltas and epsilons as follows (suffix notation).

$$\epsilon_{ijk}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}.$$

Intuition: What is the rank of a tensor?

Key idea: The number of free indices determines the **rank** (or order) of a tensor.

- 0 free indices $\Rightarrow$ rank-0 tensor (a **scalar**);
- 1 free index $\Rightarrow$ rank-1 tensor (a **vector**);
- 2 free indices $\Rightarrow$ rank-2 tensor (a **matrix**);
- $\dots$

Important remark

This is only an **intuition**. Later we will see that an object must satisfy additional transformation properties to genuinely qualify as a tensor. So this should not be taken as a formal definition of rank.

Vector differential operators

Definition

- A **scalar field** is a map that assigns a **real number** to every point in space.

$$\phi : \mathbb{R}^n \to \mathbb{R}$$

- A **vector field** is a map that assigns a **vector** to every point in space.

$$\psi : \mathbb{R}^n \to \mathbb{R}^n$$

Differential Operators

We will consider three differential operators:

- The gradient
- The divergence
- The curl

Each can be expressed in suffix notation to give more compact formulations and easier calculations.

To do this we re-label the Cartesian coordinate system (x, y, z) as

$$(x_1, x_2, x_3).$$

Definition.

The **gradient** of a scalar field is

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right).$$

Remark.

- The i-th component of the gradient is the partial derivative with respect to x_i.
- So in suffix notation we can simply write

$$[\nabla f]_i = \frac{\partial f}{\partial x_i}.$$

- The gradient of a scalar has one **free index** i, indicating the result is a **vector quantity**.

Definition.

The **divergence** of a vector field $\mathbf{u}$ is

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_j}{\partial x_j} = \nabla_i u_i,$$

Remark.

- There is a dummy index j indicating the summation over $j = 1, 2, 3$.
- **No free indices** indicates the divergence of a vector is a **scalar quantity**.

Definition.

The curl of a vector field $\mathbf{u}$ is

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

Reminder: Cross product and alternating tensor

Because the **curl** uses the **cross product**, we can write it using the alternating tensor.

Reminder: alternating tensor

The **alternating tensor** ϵ_{ijk} is defined by

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i, j, k \text{ are equal,} \\ +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (2,1,3) \text{ or } (3,2,1). \end{cases}$$

Reminder: Alternating tensor and cross product

Recall that

$$(\mathbf{u} \times \mathbf{v})_i = \epsilon_{ijk} u_j v_k.$$

Curl and cross product

The components of the curl are then

$$[\nabla \times \mathbf{u}]_i = \epsilon_{ijk} \nabla_j u_k = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}.$$

Remark.

- The dummy indices j and k indicate a double sum.
- The free index i indicates the curl of a vector is a **vector quantity**.

Example: The Curl

Let us find the curl of

$$\mathbf{A} = \frac{1}{2}x_1^3 x_2 \mathbf{i} + (4x_1 x_2^5 + 1)\mathbf{j} + x_2 x_3^3 \mathbf{k}$$

at the point $Q = (2, 1, 2)$.

$$\begin{aligned}
\nabla \times \mathbf{A} &= \left[\frac{\partial}{\partial x_2}\left(x_2 x_3^3\right) - \frac{\partial}{\partial x_3}\left(4x_1 x_2^5 + 1\right)\right]\mathbf{i} \\
&\quad - \left[\frac{\partial}{\partial x_1}\left(x_2 x_3^3\right) - \frac{\partial}{\partial x_3}\left(\frac{1}{2}x_1^3 x_2\right)\right]\mathbf{j} \\
&\quad + \left[\frac{\partial}{\partial x_1}\left(4x_1 x_2^5 + 1\right) - \frac{\partial}{\partial x_2}\left(\frac{1}{2}x_1^3 x_2\right)\right]\mathbf{k} \\
&= x_3^3 \mathbf{i} + 0\mathbf{j} + \left(4x_2^5 - \frac{1}{2}x_1^3\right)\mathbf{k}.
\end{aligned}$$

Example: The Curl

Thus, at $Q = (2,1,2)$ we have

$$\begin{aligned}\nabla \times \mathbf{A}\big|_{(2,1,2)} &= \left[x_3^3\mathbf{i} + 0\mathbf{j} + \left(4x_2^5 - \frac{1}{2}x_1^3\right)\mathbf{k}\right]_{(2,1,2)} \\ &= 8\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} \\ &= (8,0,0).\end{aligned}$$

Vector differential operators

Position vector

Definition.

The **position vector** is

$$\mathbf{r} = (x_1, x_2, x_3).$$

We denote its **magnitude** by

$$\underline{r} = |\mathbf{r}|.$$

Next, let us find

- the gradient $\nabla \underline{r}$,
- the divergence $\nabla \cdot \mathbf{r}$, and
- the curl $\nabla \times \mathbf{r}$ of the position vector.

Position vector derivatives and the delta Kronecker

Position vector derivatives and the delta Kronecker

- Notice that
$$\frac{\partial x_1}{\partial x_1} = 1 \quad \text{but} \quad \frac{\partial x_1}{\partial x_2} = 0.$$
- In general, we have
$$\frac{\partial x_i}{\partial x_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} = \delta_{ij}.$$

The gradient ∇r

Let $\mathbf{r}$ be the position vector $\mathbf{r} = (x_1, x_2, x_3)$ and $\underline{r} = |\mathbf{r}|$.

- We first write
$$\underline{r} = |\mathbf{r}| = (\mathbf{r}\cdot\mathbf{r})^{1/2} = (x_j x_j)^{1/2}.$$
- Then we use this to calculate $\nabla \underline{r}$

$$\begin{aligned}
[\nabla \underline{r}]_i &= \frac{\partial}{\partial x_i}(x_j x_j)^{1/2} \\
&= \frac{1}{2}(x_j x_j)^{-1/2}\frac{\partial}{\partial x_i}(x_j x_j) = \frac{1}{2\underline{r}}\frac{\partial}{\partial x_i}(x_j x_j) \qquad \text{(chain rule)} \\
&= \frac{1}{2\underline{r}} 2x_j \frac{\partial x_j}{\partial x_i} \qquad \text{(product rule)} \\
&= \frac{1}{\underline{r}} x_j \delta_{ij} = \frac{x_i}{\underline{r}} = \left[\frac{\mathbf{r}}{\underline{r}}\right]_i .
\end{aligned}$$

Divergence of vector position

The divergence $\nabla \cdot \mathbf{r}$

Let $\mathbf{r}$ be the position vector $\mathbf{r} = (x_1, x_2, x_3)$.

Let us find the divergence using suffix notation

$$\nabla \cdot \mathbf{r} = \frac{\partial r_i}{\partial x_i} = \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3.$$

Curl of vector position

The curl $\nabla \times \mathbf{r}$

Let $\mathbf{r}$ be the position vector $\mathbf{r} = (x_1, x_2, x_3)$

Let us find the curl

$$[\nabla \times \mathbf{r}]_i = \epsilon_{ijk} \frac{\partial x_k}{\partial x_j} = \epsilon_{ijk} \delta_{jk} = (\mathbf{0})_i,$$

where $\mathbf{0} = (0, 0, 0)$.

In fact, $\epsilon_{ijk} \delta_{jk} = 0$ because

- If $j \neq k$, then $\delta_{jk} = 0$.
- If $j = k$, then $\epsilon_{ijk} = 0$.

This means that $\nabla \times \mathbf{r} = (0, 0, 0)$.

Combinations of grad, div and curl

Combining Differential Operators

We will consider the following combinations of the differential operators

1. Div grad,
2. Curl grad,
3. Grad div,
4. Div curl,
5. Curl curl.

We will use **suffix notation** because it is much quicker than writing out all the components.

Div grad

We will first compute the combination **div grad**.

But first, note that this is well defined because

$$\begin{aligned} \text{grad} &: \text{scalar field} \to \text{vector field}, \\ \text{div} &: \text{vector field} \to \text{scalar field}. \end{aligned}$$

Hence

$$\text{div} \circ \text{grad} : \text{scalar field} \xrightarrow{\text{grad}} \text{vector field} \xrightarrow{\text{div}} \text{scalar field}.$$

In other words, div grad is an operator of the form

$$\text{scalar field} \to \text{scalar field}.$$

Div grad

Consider scalar field f. The **div grad** of f is

$$\begin{aligned}\nabla \cdot (\nabla f) &= \frac{\partial}{\partial x_j} \left([\nabla f]_j \right) \\ &= \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_j} \right) \\ &= \frac{\partial^2 f}{\partial x_j \partial x_j} = \nabla^2 f.\end{aligned}$$

The symbol ∇^2 is known as the **Laplacian operator**.

Curl grad

Let us compute **curl grad**. Consider scalar field f. We have

$$
\begin{aligned}
[\nabla \times (\nabla f)]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k} && ((\mathbf{w} \times \mathbf{v})_i = \epsilon_{ijk} w_j v_k) \\
&= \epsilon_{ikj} \frac{\partial}{\partial x_k} \frac{\partial f}{\partial x_j} && (\text{relabelling } j \leftrightarrow k) \\
&= -\epsilon_{ijk} \frac{\partial}{\partial x_k} \frac{\partial f}{\partial x_j} && (\epsilon_{ikj} = -\epsilon_{ijk}) \\
&= -\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k} && (\text{order of derivatives does not matter}).
\end{aligned}
$$

Curl grad

Let us compute **curl grad**. Consider scalar field f. We have

$$
\begin{aligned}
[\nabla \times (\nabla f)]_i &= gray\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} && ((\mathbf{w}\times\mathbf{v})_i = \epsilon_{ijk}w_j v_k) \\
&= \epsilon_{ikj}\frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} && (\text{relabelling } j \leftrightarrow k) \\
&= -\epsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} && (\epsilon_{ikj} = -\epsilon_{ijk}) \\
&= gray{-}\epsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} && (\text{order of derivatives does not matter}).
\end{aligned}
$$

Curl grad - conclusion

Curl grad

Because

$$[\nabla \times (\nabla f)]_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k} = -\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial f}{\partial x_k},$$

we conclude

$$[\nabla \times (\nabla f)]_i = (\mathbf{0})_i,$$

that is

$$\nabla \times (\nabla f) = \mathbf{0} = (0, 0, 0).$$

Grad div

Let us compute **grad div**. Consider vector field $\mathbf{u}$. We have

$$[\nabla(\nabla\cdot\mathbf{u})]_i = \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) = \frac{\partial^2 u_j}{\partial x_i \partial x_j}.$$

This quantity cannot be simplified any further.

Div curl

Let us compute **div curl**. Consider a vector field $\mathbf{u}$. We have

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{u}) &= \nabla_i (\nabla \times \mathbf{u})_i & (\mathbf{w} \cdot \mathbf{v} = w_i v_i) \\
&= \nabla_i \left(\epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \right) & ((\mathbf{w} \times \mathbf{v})_i = \epsilon_{ijk} w_j v_k) \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_j} & \left(\nabla_i = \frac{\partial}{\partial x_i} \right) \\
&= \epsilon_{jik} \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_i} & (\text{relabelling } i \leftrightarrow j) \\
&= -\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_i} & (\epsilon_{jik} = -\epsilon_{ijk}) \\
&= -\epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_j}. & (\text{order of derivatives does not matter})
\end{aligned}$$

Div curl

Let us compute **div curl**. Consider a vector field $\mathbf{u}$. We have

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{u}) &= \nabla_i (\nabla \times \mathbf{u})_i && (\mathbf{w} \cdot \mathbf{v} = w_i v_i) \\
&= \nabla_i \left(\epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \right) && ((\mathbf{w} \times \mathbf{v})_i = \epsilon_{ijk} w_j v_k) \\
&= gray \epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_j} && \left(\nabla_i = \frac{\partial}{\partial x_i} \right) \\
&= \epsilon_{jik} \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_i} && (\text{relabelling } i \leftrightarrow j) \\
&= -\epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial u_k}{\partial x_i} && (\epsilon_{jik} = -\epsilon_{ijk}) \\
&= gray - \epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_j}. && (\text{order of derivatives does not matter})
\end{aligned}$$

Curl grad

Because

$$\nabla \cdot (\nabla \times \mathbf{u}) = \epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_j} = -\epsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial u_k}{\partial x_j},$$

we conclude

$$\nabla \cdot (\nabla \times \mathbf{u}) = \mathbf{0}.$$

Curl curl

Let us compute **curl curl**. Consider vector field $\mathbf{u}$. We have

$$[\nabla \times (\nabla \times \mathbf{u})]_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} (\nabla \times \mathbf{u})_k \qquad ((\mathbf{w} \times \mathbf{v})_i = \epsilon_{ijk} w_j v_k)$$

$$= \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{k\ell m} \frac{\partial u_m}{\partial x_\ell}$$

$$= \epsilon_{ijk} \epsilon_{k\ell m} \frac{\partial^2 u_m}{\partial x_j \partial x_\ell}$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) \frac{\partial^2 u_m}{\partial x_j \partial x_\ell}$$

$$= \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$= [\nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}]_i.$$

curl curl & the Laplacian

We have shown

$$[\nabla \times (\nabla \times \mathbf{u})]_i = [\nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}]_i.$$

That is,

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}.$$

This gives us a new description of the vector Laplacian operator.

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}).$$

Example

We now use the new description of the Laplacian

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}),$$

to show that

$$\nabla \times (\nabla^2 \mathbf{u}) = \nabla^2(\nabla \times \mathbf{u}).$$

First substitute in the identity:

$$\nabla \times (\nabla^2 \mathbf{u}) = \nabla \times \big(\nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})\big).$$

Example - Part 2

Example

Next, we show that

$$\nabla \times (\nabla^2 \mathbf{u}) = \nabla^2 (\nabla \times \mathbf{u}).$$

Using properties of combinations, we get

$$\begin{aligned}\nabla \times (\nabla^2 \mathbf{u}) &= \nabla \times \big(\nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})\big) \\ &= -\nabla \times (\nabla \times (\nabla \times \mathbf{u})) \quad \text{(since curl grad is zero)}\end{aligned}$$

We know that curl curl is

$$\nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u}.$$

Substituting this in the formula above, we get

$$-\nabla(\nabla \cdot (\nabla \times \mathbf{u})) + \nabla^2 (\nabla \times \mathbf{u}) = \nabla^2 (\nabla \times \mathbf{u}) \quad \text{(div curl is zero)}.$$

Remark

Notice that we have shown

$$\nabla \times (\nabla^2 \mathbf{u}) = \nabla^2 (\nabla \times \mathbf{u}).$$

That is, the operators ∇^2 and $\nabla\times$ commute.

Summary: Combinations of grad, div, curl

1. Div Grad:
$$\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x_j \partial x_j} = \nabla^2 f,$$
2. Curl Grad:
$$\nabla \times (\nabla f) = \mathbf{0},$$
3. Grad Div:
$$[\nabla(\nabla \cdot \mathbf{u})]_i = \frac{\partial^2 u_j}{\partial x_i \partial x_j},$$
4. Div Curl:
$$\nabla \cdot (\nabla \times \mathbf{u}) = \mathbf{0},$$
5. Curl Curl:
$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}).$$

Next time . . .

Next time...

- Grad, div and curl applied to functions.