Tensor Analysis

Slides week 23 – Lecture 2

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Reminder

Portfolio

- The **Portfolio Test** will take place on 11 March, 2026.
  - Duration: 1:15 minutes (+25 minutes if you are entitle to extra time).
  - Main Cohort: INB2101
  - PASS or Reasonable adjustments: DCB1105
- The content covers everything we have learned up to the end of this week.

Portfolio overview

Part A — Take-home (25%)

- Released **this Wednesday**.
- Complete at your own pace at home.
- You must **bring Part A** with you to the TCA.

Part B — Time Constrained Assessment (TCA) (75%)

- Takes place on **11 March 2026**

Submission

- Submit **Part A and Part B together** as a **single PDF file**.
- If you cannot submit both together, **e-mail part A** to me.

Chapter 4: Tensors

Today: Chapter 4–Tensors

1. Preliminaries,
2. The quotient rule,
3. Symmetric and antisymmetric tensors.

Reminder

Definition: Tensor

Definition

In **orthogonal coordinate system**, a quantity is a **tensor** if each of the free suffices transform in a certain way under rotation of the coordinates.

For instance,

- **Rank 2:** $T'_{ij} = L_{im}L_{jn}T_{mn}$,
- **Rank 3:** $T'_{ijk} = L_{im}L_{jn}L_{kp}T_{mnp}$,
- **Rank 4:** $T'_{ijk\ell} = L_{im}L_{jn}L_{kp}L_{\ell q}T_{mnpq}$, etc.

Note that on the RHS m, n, p and q are repeated dummy indices (summed over), and i, j, k, and ℓ are free indices.

Thus, the equation balances.

Definition: Rank

Definition

The **rank** or **order** of a tensor is the number of free indices.

Example

A 3×3 matrix $\mathbf{M}$ is written as M_{ij} with two free indices i and j which act as row and column counters.

Thus, M_{ij} is a **second-rank** tensor.

Remark

A tensor may have any number of free indices.

For instance, a rank 7 tensor $P_{ijk\ell mnp}$ transforms via:

$$P'_{ijk\ell mnp} = L_{ia}L_{jb}L_{kc}L_{\ell d}L_{me}L_{nf}L_{pg}P_{abcdefg}.$$

Example 1: Kronecker Delta

The Kronecker Delta δ_{ij} is a **second-rank** tensor.

To show this, we must show

$$\delta'_{ij} = L_{ik} L_{jm} \delta_{km}.$$

In fact, recall that $L_{ik} L_{jk} = \delta_{ij}$. Thus,

$$L_{ik} L_{jm} \delta_{km} = L_{ik} L_{jk} = \delta_{ij}.$$

Since δ_{ij} is defined in the same in any coordinate system, $\delta_{ij} = \delta'_{ij}$. Thus,

$$L_{ik} L_{jm} \delta_{km} = \delta'_{ij}.$$

Example 2: The Gradient of a Vector

Show that $\nabla \mathbf{u} = \frac{\partial u_i}{\partial x_j}$ is a second-rank tensor.

We must show

$$\frac{\partial u_i'}{\partial x_j'} = L_{ik} L_{j\ell} \frac{\partial u_k}{\partial x_\ell}.$$

Recall that

$$u_i' = L_{ik} u_k \quad \text{and} \quad \frac{\partial x_i}{\partial x_j'} = L_{ji}.$$

Thus,

$$\frac{\partial u_i'}{\partial x_j'} = \frac{\partial L_{ik} u_k}{\partial x_j'} = \underbrace{L_{ik} \frac{\partial u_k}{\partial x_j'} + u_k \frac{\partial L_{ik}}{\partial x_j'}}_{\text{product rule}}.$$

Remark

Recall that

$$\frac{\partial x'_i}{\partial x_k} = L_{ik} \qquad \text{and} \qquad \frac{\partial x_i}{\partial x'_k} = L_{ki}.$$

Thus,

$$\begin{aligned}\frac{\partial L_{ik}}{\partial x'_j} &= \frac{\partial}{\partial x'_j}\left(\frac{\partial x'_i}{\partial x_k}\right) = \frac{\partial^2 x'_i}{\partial x'_j \partial x_k} \\ &= \frac{\partial^2 x'_i}{\partial x_k \partial x'_j} = \frac{\partial}{\partial x_k}\left(\frac{\partial x'_i}{\partial x'_j}\right). \\ &= \frac{\partial}{\partial x_k}\left(\delta_{ij}\right) \\ &= 0.\end{aligned}$$

Example 2: The Gradient of a Vector

Show that $\nabla \mathbf{u} = \frac{\partial u_i}{\partial x_j}$ is a second-rank tensor.

We must show

$$\frac{\partial u_i'}{\partial x_j'} = L_{ik} L_{j\ell} \frac{\partial u_k}{\partial x_\ell}.$$

So far, we have

$$\frac{\partial u_i'}{\partial x_j'} = L_{ik} \frac{\partial u_k}{\partial x_j'} + u_k \underbrace{\frac{\partial L_{ik}}{\partial x_j'}}_{=0} = L_{ik} \frac{\partial u_k}{\partial x_j'}$$

$$= L_{ik} \underbrace{\frac{\partial u_k}{\partial x_\ell} \frac{\partial x_\ell}{\partial x_j'}}_{\text{chain rule}} = L_{ik} \underbrace{\frac{\partial u_k}{\partial x_\ell} L_{j\ell}}_{\frac{\partial x_\ell}{\partial x_j'} = L_{j\ell}} = L_{ik} L_{j\ell} \frac{\partial u_k}{\partial x_\ell}.$$

Your turn!

Given that **a** and **b** are vectors, show that the quantity $a_i b_j$ is a second-rank tensor.

Practical question

If T_{ij} is a tensor, show that T_{ii} is a scalar.

Your turn!

Suppose that **c** and **d** are vectors. Show that their cross product

$$\mathbf{c} \times \mathbf{d}$$

is a vector.

[You can use the fact that the alternating tensor ϵ_{ijk} is a tensor.]

The quotient rule

The quotient rule

Lemma (The quotient rule)

Suppose there is a quantity T_{ij} such that in **all rotated coordinate systems**, we have that for **any** vector $\mathbf{b}$,

$$T_{ij}b_j$$

is a vector.

In other words, the resulting quantity $\mathbf{a}$ given by

$$a_i = T_{ij}b_j$$

is a vector.

Then T_{ij} is a tensor.

Proof.

Assumptions:

A1. **b** is a vector i.e. $b'_m = L_{mj} b_j \quad \text{or} \quad b_j = L_{mj} b'_m$,

A2. We can write $a_k = T_{kj} b_j$,

A3. **a** is a vector i.e. $a'_i = L_{ik} a_k$.

We want to show: T_{ij} is a tensor i.e.

$$T'_{ij} = L_{ik} L_{jm} T_{km}.$$

Proof.

Assumptions:

A1. $\mathbf{b}$ is a vector i.e. $b'_m = L_{mj} b_j \quad \text{or} \quad b_j = L_{mj} b'_m$,

A2. We can write $a_k = T_{kj} b_j$,

A3. $\mathbf{a}$ is a vector i.e. $a'_i = L_{ik} a_k$.

On the one hand,

$$\begin{aligned} a'_i = L_{ik} a_k &= L_{ik} T_{kj} b_j && \text{(A3 \& A2)} \\ &= L_{ik} T_{kj} L_{mj} b'_m && \text{(A1)} \\ &= L_{ik} L_{mj} T_{kj} b'_m. \end{aligned}$$

On the other hand,

$$a'_i = T'_{im} b'_m. \qquad \text{(A2 holds in all coordinate systems)}$$

Proof.

Thus, we conclude

$$0 = a'_i - a'_i = T'_{im} b'_m - L_{ik} L_{mj} T_{kj} b'_m = (T'_{im} - L_{ik} L_{mj} T_{kj}) b'_m.$$

Because **b** can be any vector, we can assume $b'_m \neq 0$. We then obtain

$$T'_{im} - L_{ik} L_{mj} T_{kj} = 0.$$

That is,

$$T'_{im} = L_{ik} L_{mj} T_{kj}.$$

Therefore, T_{ij} is a second-rank tensor. □

Symmetric and Antisymmetric Tensors

Definition: Symmetric and antisymmetric tensors

Definition.

A second-rank tensor T_{ij} is **symmetric** if

$$T_{ij} = T_{ji}.$$

Definition.

A second-rank tensor T_{ij} is **antisymmetric** if

$$T_{ij} = -T_{ji}.$$

Example.

We know that the Kronecker delta is symmetric:

$$\delta_{ij} = \delta_{ji}.$$

Definition: Symmetric and antisymmetric tensors - higher rank

Definition.

A tensor of rank greater than two can be symmetric or antisymmetric with respect to **any pair of indices**.

Example.

The alternating tensor ϵ_{ijk} is antisymmetric:

$$\epsilon_{ijk} = -\epsilon_{jik},$$
$$\epsilon_{ijk} = -\epsilon_{ikj}, \text{ and}$$
$$\epsilon_{ijk} = -\epsilon_{kji}.$$

Symmetry Lemma

Lemma.

Symmetry is a **physical property** of tensors.

In other words, if a tensor is symmetric in a Cartesian coordinate system then it is also symmetric in **any** other Cartesian coordinate systems.

Proof.

Proof Suppose that A_{ij} is a symmetric tensor. That is, $A_{ij} = A_{ji}$.

Then in a rotated frame, we have

$$A'_{ij} = L_{ik}L_{jm}A_{km} = L_{ik}L_{jm}\underbrace{A_{mk}}_{\text{symmetric}} = \underbrace{L_{jm}L_{ik}}_{\text{re-order}}A_{mk} = A'_{ji}.$$

Thus, A'_{ij} is also symmetric. □

Example 1

Show that any second-rank tensor T_{ij} can be written as the sum of a symmetric tensor and an antisymmetric tensor.

For any tensor T_{ij},

- $T_{ij} + T_{ji}$ is symmetric,
- $T_{ij} - T_{ji}$ is antisymmetric.

We write T_{ij} using sums of these symmetric and anti-symmetric tensors:

$$T_{ik} = \underbrace{\frac{1}{2}(T_{ik} + T_{ki})}_{S_{ik}} + \underbrace{\frac{1}{2}(T_{ik} - T_{ki})}_{A_{ik}}.$$

Thus,

$$T_{ik} = S_{ik} + A_{ik}.$$

Symmetrisation

Notation:

- $S_{ik} = \frac{1}{2}(T_{ik} + T_{ki})$ is called the **symmetric part** of T_{ik},
- $A_{ik} = \frac{1}{2}(T_{ik} - T_{ki})$ is called the **antisymmetric part** of T_{ik}

Moreover,

- Constructing S_{ik} from an arbitrary tensor with components T_{ik} is called **symmetrisation**,
- Constructing A_{ik} from an arbitrary tensor with components T_{ik} is called **antisymmetrisation**.

Example 2

A second-rank tensor T_{ij} obeys $\epsilon_{ijk}T_{jk} = \mathbf{0}_i$.

Show that T_{ij} is a symmetric tensor.

Recall the alternating tensor

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i,j,k \text{ are equal,} \\ +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (2,1,3) \text{ or } (3,2,1). \end{cases}$$

Taking $i = 1$, we obtain

$$\epsilon_{1jk}T_{jk} = \sum_j \sum_k \epsilon_{1jk}T_{jk} = \underbrace{\epsilon_{123}T_{23} + \epsilon_{132}T_{32}}_{\text{only non-zero components}} = T_{23} - T_{32}.$$

Example 2

A second-rank tensor T_{ij} obeys $\epsilon_{ijk}T_{jk} = \mathbf{0}_i$.

Show that T_{ij} is a symmetric tensor.

Now, because $\epsilon_{ijk}T_{jk} = \mathbf{0}_i$, and $\epsilon_{ijk}T_{jk} = T_{23} - T_{32}$, we see that

$$T_{23} = T_{32}.$$

If we repeat this for $i = 2$ and $i = 3$, we get

$$T_{12} = T_{21} \quad \text{and} \quad T_{13} = T_{31}.$$

We then conclude that $T_{ij} = T_{ji}$, hence T_{ij} is symmetric.

Task

Your turn!

Why is the metric tensor $g_{jk} = \mathbf{e}_j \cdot \mathbf{e}_k$ symmetric?

Your turn!

If B_{rs} is an antisymmetric tensor, show that $B_{rr} = 0$.

Your turn!

Given two vectors **A** and **B**, with components A_i and B_i, show that the second-order tensor with components

$$C_{ik} = A_i B_k - A_k B_i$$

is antisymmetric.

Next Lecture

Next time...

- Chapter 5: Local Coordinate Transform
  - Associated tensors,
  - The metric tensor,
  - Higher order tensors in generalised coordinates.