Tensor Analysis

Slides week 24 – Lecture 2

Paula Lins

University of Lincoln

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Reminder

Portfolio

- The **Portfolio Test** will take place on 11 March, 2026.
  - Duration: 1:15 minutes (+25 minutes if you are entitle to extra time).
  - Main Cohort: INB2101
  - PASS or Reasonable adjustments: DCB1105
- The content covers everything we have learned up to the end of this week.

Portfolio overview

Part A — Take-home (25%)

- Questions are already available.
- Complete at your own pace at home.
- You must **bring Part A** with you to the TCA.

Part B — Time Constrained Assessment (TCA) (75%)

- Takes place this week on **11 March 2026**

Submission

- Submit **Part A and Part B together** as a **single PDF file**.
- If you cannot submit both together, **e-mail part A** to me.

Chapter 5: Tensors in generalised coordinate systems

Today: Chapter 5– Tensors in generalised coordinate systems

1. Covariant, contravariant and mixed components of a tensor,
2. Associated tensors,
3. The metric tensor,
4. Higher order tensors in generalised coordinates.

Reminder

Covariant and contravariant - first-rank tensors

In a generalised coordinate system in 3D, a first-rank tensor (i.e. a vector) $\mathbf{A}$ is uniquely determined either by

- its three covariant components A_i,
- its three contravariant components A^i

Covariant and contravariant components are related:

$$A_i = g_{ik} A^k,$$
$$A^i = g^{ik} A_k$$

where g_{ik} and g^{ik} are determined by the basis of the coordinate system .

Second-rank tensors - covariant and contravariant components

Second-rank tensors - covariant and contravariant components

In dimension 3D, a second rank tensor is specified by its **nine** components, which can be

- **covariant** A_{ik},
- **contravariant** A^{ik},
- **mixed** $A_{i}^{.k}$, $A^{i}_{.k}$.

Second-rank tensors - Notation of mixed components

Notation

Notice that the mixed components are denoted with dots:

$$A_i^{\cdot k} \quad \text{and} \quad A^i_{\cdot k}.$$

The dot in the mixed components emphasizes the order of occurrence of the indices.

For instance, in $A_i^{\cdot k}$, the first index is "covariant" and the second "contravariant":

$$\text{“}A_i^{\cdot k} = (A_i)^k\text{”}$$

while in $A^i_{\cdot k}$, the first index is "contravariant" and the "second covariant"':

$$\text{“}A^i_{\cdot k} = (A^i)_k\text{”}.$$

Second-rank tensors - transformations

The components are transformed via:

$$A'_{ik} = L^{\ell}_{i'} L^{m}_{k'} A_{\ell m},$$
$$A'^{ik} = L^{i'}_{\ell} L^{k'}_{m} A^{\ell m},$$
$$A'^{\cdot k}_{i} = L^{\ell}_{i'} L^{k'}_{m} A^{\cdot m}_{\ell},$$
$$A'^{i}_{\cdot k} = L^{i'}_{\ell} L^{m}_{k'} A^{\ell}_{\cdot m},$$

where $L^{k}_{i'}$ and $L^{k'}_{i}$ $(i, k = 1, 2, 3)$ are the coefficients of the direct and the inverse transformations.

Second-rank tensors - relations between components

Second-rank tensors - relations between components

The relations between the various components of a tensor are:

$$\begin{aligned}
A_{ik} &= g_{i\ell} g_{km} A^{\ell m} = g_{k\ell} A_i^{\cdot\ell} = g_{i\ell} A^{\ell}_{\cdot k},\\
A^{ik} &= g^{i\ell} g^{km} A_{\ell m} = g^{i\ell} A_{\ell}^{\cdot k} = g^{k\ell} A^{i}_{\cdot\ell},\\
A_i^{\cdot k} &= g^{k\ell} A_{i\ell} = g_{i\ell} A^{\ell k},\\
A^{i}_{\cdot k} &= g^{i\ell} A_{\ell k} = g_{k\ell} A^{i\ell}.
\end{aligned}$$

Second-rank tensors – Symmetry and Antisymmetry

Second-rank tensors – Symmetry and Antisymmetry

Symmetry and antisymmetry apply only to pairs of indices in the same positions:

- $A_{ik}^{\cdot\cdot\ell}$ is **symmetric** in i and k if

$$A_{ik}^{\cdot\cdot\ell} = A_{ki}^{\cdot\cdot\ell}.$$

- $B^{ik}_{\cdot\cdot\ell}$ is **anti-symmetric** in i and k if

$$B^{ik}_{\cdot\cdot\ell} = -B^{ki}_{\cdot\cdot\ell}.$$

Indices in different positions

Last week, we showed that if

- A_i is a covariant tensorand
- B^i is a contravariant tensor,

then $A_i + B^i$ does not form a tensor.

Associated tensors

Lowering indices

Lowering indices

Given a tensor, we can derive other tensors by **raising or lowering indices**.

For instance, given the tensor A_{pq}, by raising the index p, we obtain the tensor $A^{p}_{.q}$.

(The dot indicates the original position of the moved index.)

By raising the index q as well, we obtain $A^{pq}_{..}$.

Remark:

Sometimes the dots might be omitted.

Deriving new tensors

These derived tensors can be obtained by forming inner products of the given tensor with the metric tensor g_{pq} or g^{pq}:

$$A^{p}_{\cdot q} = g^{rp} A_{rq} \qquad \text{and } A_{p}^{\cdot q} = g_{rp} A^{rq}$$

Rule:

- On (both) RHS: look for repeated index (in this case r).
- We then replace this index according to the metric tensor:
  - g^{rp} and g_{rp} both mean we replace r with p.
- Raise or lower the index according to the metric tensor:
  - g^{rp} means we raise p, while
  - g_{rp} means we lower p.

Deriving new tensors - Second Example

Another example is

$$A^{pq} = g^{rp} g^{sq} A_{rs}.$$

Again, we use the rules:

- On RHS: look for repeated indices (in this case r and s).
- Replace these indices according to the metric tensor:
  - g^{rp} means replace r with p
  - and g^{sq} means we replace s with q.
- Raise or lower the index according to the metric tensor:
  - g^{rp} and g^{sq} mean we raise p and q, respectively.

Deriving new tensors - Third Example

Deriving new tensors - Third Example

We also have

$$A^{p}_{\cdot rs} = g_{rq} A^{pq}_{\cdot\cdot s}$$

- On RHS: look for repeated index (in this case q).
- Replace this index according to the metric tensor:
  - g_{rq} means we replace q with r.

  Raise or lower the index according to the metric tensor:
  - g_{rq} means we lower r.

Deriving new tensors - Last Example

The following holds:

$$A^{qm.tk}_{\cdot\cdot n} = g^{pk} g_{sn} g^{rm} A^{q\cdot st}_{\cdot r\cdot\cdot p}$$

Rules:

- On RHS: look for repeated indices (in this case p, r, and s).
- Replace these indices according to metric tensors:
  - g^{pk} means replace p with k,
  - g_{sn} means replace s with n, and
  - g^{rm} means replace r with m.
- Raise or lower the index according to the position of the indices in the metric tensor:
  - g^{pk} and g^{sq} mean we raise k and q, respectively,
  - g_{sn} means we lower n.

Associated tensors

Associated tensors

All tensors obtained as in the previous slides (by lowering or raising indices) are called **associated tensors** of the given tensor.

Example

A^p and A_p are associated tensors. The relation between them is given by

$$A_p = g_{pq} A^q \quad \text{or} \quad A^p = g^{pq} A_q.$$

Associated tensors - Cartesian Coordinates

Associated tensors - Cartesian Coordinates

For Cartesian coordinates,

$$g_{pq} = \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{if } p \neq q. \end{cases}$$

As we have A^p and A_p are associated tensors with relation given by

$$A_p = g_{pq} A^q \quad \text{or} \quad A^p = g^{pq} A_q.$$

We see that in a Coordinate system, $A_p = A^p$.

This explains why no distinction was made between contravariant and covariant components of a vector earlier.

The metric tensor

Metric Tensor as components of a tensor

Metric Tensor as components of a tensor

The quantities g_{ik}, g^{ik}, $g_i^{\cdot k}$ are actually the components of a second-rank tensor, called the **metric tensor**.

Let us show that g_{ik} is in fact a second-rank tensor. By defintion

$$g'_{ik} = \mathbf{e}'_i \cdot \mathbf{e}'_k = (L^{\ell}_{i'}\mathbf{e}_\ell) \cdot (L^{m}_{k'}\mathbf{e}_m) \\ = L^{\ell}_{i'} L^{m}_{k'} (\mathbf{e}_\ell \cdot \mathbf{e}_m) = L^{\ell}_{i'} L^{m}_{k'} g_{\ell m}.$$

We obtained that

$$g'_{ik} = L^{\ell}_{i'} L^{m}_{k'} g_{\ell m}$$

that is, g_{ik} is a tensor of rank 2.

Metric Tensor as components of a tensor - Part 2

Metric Tensor as components of a tensor

Similarly,

$$g'^{ik} = \mathbf{e}'^{i} \cdot \mathbf{e}'^{k} = L^{i'}_{\ell} L^{k'}_{m} (\mathbf{e}^{\ell} \cdot \mathbf{e}^{m}) = L^{i'}_{\ell} L^{k'}_{m} g^{\ell m},$$

$$g'^{\cdot k}_{i} = \mathbf{e}'_{i} \cdot \mathbf{e}'^{k} = L^{\ell}_{i'} L^{k'}_{m} (\mathbf{e}_{\ell} \cdot \mathbf{e}^{m}) = L^{\ell}_{i'} L^{k'}_{m} g^{\cdot m}_{\ell}.$$

We then see that

- g_{ik} are the covariant components of some tensor,
- g^{ik} are contravariant components of some tensor,
- $g^{\cdot k}_{i}$ are mixed components of some tensor.

Metric Tensor as components of a tensor - Part 3

Question

Are g_{ik}, g^{ik}, and $g_i^{\cdot k}$ components of the same tensor?

Let us check

To verify that all these quantities are components of the **same** tensor, we need only show that they are connected by relations of the form

$$A_{ik} = g_{i\ell} g_{km} A^{\ell m} = g_{k\ell} A_i^{\cdot \ell} = g_{i\ell} A^{\ell}_{\cdot k},$$
$$A^{ik} = g^{i\ell} g^{km} A_{\ell m} = g^{i\ell} A_\ell^{\cdot k} = g^{k\ell} A^{i}_{\cdot \ell},$$
$$A_i^{\cdot k} = g^{k\ell} A_{i\ell} = g_{i\ell} A^{\ell k},$$
$$A^{i}_{\cdot k} = g^{i\ell} A_{\ell k} = g_{k\ell} A^{i\ell}.$$

Metric Tensor as components of a tensor - Part 4

Metric Tensor as components of the same tensor

It follows from the definition of g_{ik} and the properties of the basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and its dual $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ that

$$\mathbf{e}_i = g_{i\ell}\mathbf{e}^{\ell}.$$

Therefore

$$g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k = (g_{i\ell}\mathbf{e}^{\ell}) \cdot (g_{km}\mathbf{e}^{m}) = g_{i\ell}g_{km}(\mathbf{e}^{\ell} \cdot \mathbf{e}^{m}) = g_{i\ell}g_{km}g^{\ell m},$$
$$g_{ik} = \mathbf{e}_i \cdot \mathbf{e}_k = (g_{i\ell}\mathbf{e}^{\ell}) \cdot \mathbf{e}_k = g_{i\ell}g^{\ell}_{\cdot k},$$

as required.

Higher order tensors in generalised coordinates

Higher order tensors in generalised coordinates

Higher order tensors in generalised coordinates

- In a generalised coordinate system, a tensor of rank n has 3^n components.
- However, it can have various kinds of components. That is, covariant, contravariant and mixed components.

Components of higher order tensors in generalised coordinates

Example

A third-rank tensor has $3^3 = 27$ components. But it has mixed components $A_{ik}^{\cdot\cdot\ell}$, $A_{\cdot k\ell}^{i\cdot\cdot}$, ... which transform according to the formulae

$$A'^{\cdot\cdot\ell}_{ik} = L_{i'}^{m} L_{k'}^{n} L_{r}^{\ell'} A_{mn}^{\cdot\cdot r},$$

$$A'^{i\cdot\ell}_{\cdot k} = L_{m}^{i'} L_{k'}^{n} L_{r}^{\ell'} A_{\cdot n}^{m\cdot r}, \text{ etc.}$$

Here we say that $A_{ik}^{\cdot\cdot\ell}$ is a mixed tensor with two covariant indices and one contravariant index (and similarly for $A_{\cdot k}^{i\cdot\ell}$).

Next Lecture

Next time...

- Chapter 6: Tensor Algebra
  - Addition of tensors,
  - Multiplication of tensors,
  - Contraction of Tensors.