Tensor Analysis

Slides week 25 – Lecture 1

Paula Lins

University of Lincoln

2025/26

Tensor Algebra

Chapter 6: Tensor Algebra

1. Addition of tensors,
2. Multiplication of tensors,
3. Contraction of tensors.

Introduction

Products of tensors so far

We already know some types of products related to tensors. For example,

- The scalar product: $\mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$(r, \mathbf{v}) \mapsto r\mathbf{v} = (rv_1, rv_2, rv_3).$$

- The dot product: $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$(\mathbf{v}, \mathbf{u}) \mapsto \mathbf{v} \cdot \mathbf{u} = v_1u_1 + v_2u_2 + v_3u_3.$$

Products of tensors so far

We already know some types of products related to tensors. For example,

- The cross product: $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$(\mathbf{v}, \mathbf{u}) \mapsto \mathbf{v} \times \mathbf{u} = (v_2 u_3 - v_3 u_2, -(v_1 u_3 - v_3 u_1), v_1 u_2 - v_2 u_1).$$

- The matrix products:

$$\mathrm{Mat}_{m \times k}(\mathbb{R}) \times \mathrm{Mat}_{k \times n}(\mathbb{R}) \to \mathrm{Mat}_{m \times n}(\mathbb{R})$$

given by

$$(A, B) \mapsto AB.$$

Creating new tensors from existing ones

We will now explore various operations involving **tensors**:

- Addition and subtraction of tensors,
- Outer product,
- Contraction of tensors.

Each of these operations provides a method to generate new tensors from existing ones.

Addition of tensors

Addition of second rank tensors

We will now explore the sum of tensors. Let us start with the **rank two case**.

- Let A_{ik} and B_{ik} be the covariant components of two second-rank tensors.
- We can define their **sum** as $C_{ik} = A_{ik} + B_{ik}$.

However, we need to make sure that C_{ik} are the components of a covariant second-rank tensor for this to make sense.

Let us check whether this is the case.

Addition of second rank tensors

We know:

- A_{ik} and B_{ik} are the components of two second-rank tensors. Thus
  - $A'_{ik} = L^m_{i'} L^n_{k'} A_{mn}$, and $B'_{ik} = L^m_{i'} L^n_{k'} B_{mn}$.
- We need to show:
  - $C'_{ik} = L^m_{i'} L^n_{k'} C_{mn}$.

We have

$$\begin{aligned} C'_{ik} &= A'_{ik} + B'_{ik} \\ &= L^m_{i'} L^n_{k'} A_{mn} + L^m_{i'} L^n_{k'} B_{mn} \\ &= L^m_{i'} L^n_{k'} (A_{mn} + B_{mn}) \\ &= L^m_{i'} L^n_{k'} C_{mn}. \end{aligned}$$

Addition of second rank tensors

Adding two covariant second-rank tensors A_{ik} and B_{ik} results in another **covariant second-rank tensor**

$$A_{ik} + B_{ik} = C_{ik}.$$

This new tensor is called the **sum** of A_{ik} and B_{ik}.

Addition of second rank tensors

Similarly, we can show that

- Adding two rank two contravariant second-rank tensors gives a **contravariant second-rank tensor**

$$A^{ik} + B^{ik} = C^{ik}.$$

- Adding two rank two mixed second-rank tensors gives a mixed second-rank tensor:
  - $A^{i}_{\cdot k} + B^{i}_{\cdot k} = C^{i}_{\cdot k}$,
  - $A_{i}^{\cdot k} + B_{i}^{\cdot k} = C_{i}^{\cdot k}$.

.

Addition of second rank tensors - different positions

Addition of second rank tensors - different positions

Adding second-rank tensors of different structures (covariant, contravariant or mixed components) does not give a tensor!

Example

Let A_{ik} and A^{ik} be the components of covariant and of a contravariant tensor, respectively.

Then the transformation rules are

- $A'_{ik} = L^m_{i'} L^n_{k'} A_{mn}$, and
- $B'^{ik} = L^{i'}_m L^{k'}_n B^{mn}$.

Addition of second rank tensors - different positions

For the sum to be a tensor, we need that one of the following holds:

- $C_{ik} = A_{ik} + B^{ik}$ transforms according to

$$C'_{ik} = L^m_{i'} L^n_{k'} C_{mn} \qquad \text{(covariant)},$$

- $C^{ik} = A_{ik} + B^{ik}$ transforms according to

$$C^{ik\prime} = L^{i'}_m L^{k'}_n C^{mn} \qquad \text{(contravariant)},$$

- $C_i^{\cdot k} = A_{ik} + B^{ik}$ transforms according to

$$C_i^{\cdot k\prime} = L^m_{i'} L^{k'}_n C^m_{\cdot n} \qquad \text{(mixed)}$$

- $C^i_{\cdot k} = A_{ik} + B^{ik}$ transforms according to

$$C^i_{\cdot k}{}' = L^{i'}_m L^n_{k'} C^n_{\cdot m} \qquad \text{(mixed)}.$$

Addition- different positions - Part 3

Addition of second rank tensors - different positions

However

$$A'_{ik} + B'_{ik} = L^{m}_{i'} L^{n}_{k'} A_{mn} + L^{i'}_{m} L^{k'}_{n} B^{mn}$$

cannot be further simplified and it does not satisfy any of the four transformation rules.

It follows that

$$A_{ik} + B^{ik}$$

is **not** a tensor.

Addition of arbitrary rank tensors

What about tensor of **arbitrary** rank?

Addition of arbitrary rank tensors

Addition of any number of tensors of arbitrary rank is defined similarly:

$$P_{ijk} = A_{ijk} + B_{ijk} + C_{ijk}$$
$$Q^{ijk} = D^{ijk} + E^{ijk}$$
$$R_i^{\cdot jk\ell} = F_i^{\cdot jk\ell} + G_i^{\cdot jk\ell}$$
$$\text{etc.}$$

Important!

Tensors of different **ranks** and **structures** (covariant, contravariant and different types of mixed tensors) cannot be added.

Example – addition of second-rank tensors

Recall that **rank-two** tensors can be represented by matrices. Consider the following:

$$M_{ij} = \begin{pmatrix} 0 & -2 & -3 \\ 1 & 2 & 5 \\ 1 & 3 & 4 \end{pmatrix} \quad \text{and} \quad N_{ij} = \begin{pmatrix} 7 & 8 & 9 \\ 6 & 5 & 4 \\ 1 & 2 & 3 \end{pmatrix}.$$

By definition their sum is the tensor

$$C_{ij} = M_{ij} + N_{ij},$$

which is a matrix whose (i, j) component is $M_{ij} + N_{ij}$. Thus,

$$[C_{ij}] = [M_{ij} + N_{ij}] = \begin{pmatrix} 0 & -2 & -3 \\ 1 & 2 & 5 \\ 1 & 3 & 4 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 6 & 5 & 4 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 6 & 6 \\ 7 & 7 & 9 \\ 2 & 5 & 7 \end{pmatrix}.$$

Next Lecture

Next time...

- Chapter 6 - Tensor Algebra
  - Outer product,
  - Contraction of tensors.