

# TENSOR ANALYSIS

SLIDES WEEK 25 – LECTURE 2

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# TENSOR ALGEBRA

## Chapter 6: Tensor Algebra

1. Addition of tensors,
2. Multiplication of tensors,
3. Contraction of tensors.

REMINDER

REMINDER

## Addition of second rank tensors

We can add tensors of **same rank** and **same structure**.

For instance, for the **rank two case**, we have:

- Sum of covariant second-rank tensors:  $C_{ik} = A_{ik} + B_{ik}$ .
- Sum of contravariant second-rank tensors:  $C^{ik} = A^{ik} + B^{ik}$ .
- Sum of mixed second-rank tensors:
  - ▶  $C_i^{\cdot k} = A_i^{\cdot k} + B_i^{\cdot k}$ .
  - ▶  $C^i_{\cdot k} = A^i_{\cdot k} + B^i_{\cdot k}$ .

However, we showed that summing second rank tensors that do not have the same structure **does not give a tensor!**

E.g.  $R_{ik} + S^{ik}$  and  $T_i^{\cdot k} + U^i_{\cdot k}$  are **not tensors**.

## Addition of arbitrary rank tensors

Addition of any number of tensors of arbitrary rank is defined similarly:

$$\begin{aligned}P_{ijk} &= A_{ijk} + B_{ijk} + C_{ijk} \\Q^{ijk} &= D^{ijk} + E^{ijk} \\R_i^{jkl} &= F_i^{jkl} + G_i^{jkl} \\&\text{etc.}\end{aligned}$$

## Important!

Tensors of different **ranks** and **structures** (covariant, contravariant and different types of mixed tensors) **cannot** be added.

# OUTER PRODUCT OF TENSORS

## Outer product of second rank tensors

Consider two second-rank tensors with components  $A_{ik}$  and  $B_{ik}$ .

Imagine combining these tensors in the following way to form a new tensor:

$$C_{iklm} = A_{ik}B_{lm}.$$

These new components  $C_{iklm}$  form a fourth-rank tensor, known as the **(outer) product** of the original tensors  $A_{ik}$  and  $B_{ik}$ .

Let us show that this is the case.

# THE OUTER PRODUCT IS A TENSOR

The outer product is a tensor

Let us show that

$$C_{iklm} = A_{ik}B_{lm}$$

is a a fourth-rank tensor.

$A_{ik}$  and  $B_{ik}$  transform via

$$A'_{ik} = L_{i'}^n L_{k'}^p A_{np} \quad \text{and} \quad B'_{lm} = L_{l'}^r L_{m'}^s B_{rs},$$

Consequently,  $C_{iklm}$  transforms via:

$$\begin{aligned} C'_{iklm} &= A'_{ik} B'_{lm} = L_{i'}^n L_{k'}^p A_{np} L_{l'}^r L_{m'}^s B_{rs} \\ &= L_{i'}^n L_{k'}^p L_{l'}^r L_{m'}^s A_{np} B_{rs} = L_{i'}^n L_{k'}^p L_{l'}^r L_{m'}^s C_{nprs}. \end{aligned}$$

## Outer product and different structures

We have seen that tensors of **different structures** cannot be summed.

But can we take the **outer product** of tensors with different structures?

## Example

Let us examine whether the outer product

$$C_{ijk}^{\cdot\cdot\cdot\ell} = A_{ij} B_k^{\cdot\ell}$$

forms a rank-four mixed tensor, where  $A_{ij}$  and  $B_k^{\cdot\ell}$  are rank-two covariant and mixed tensors, respectively.

# OUTER PRODUCT AND DIFFERENT STRUCTURES- EXAMPLE

## Example

The transformation rules of  $A_{ij}$  and  $B_k^{\cdot\ell}$  are

- $A'_{ij} = L_{i'}^m L_{j'}^n A_{mn},$
- $B_k^{\cdot\ell} = L_{k'}^p L_q^{\ell'} B_p^{\cdot q}.$

Consequently,

$$\begin{aligned}C'_{ijk}{}^{\cdot\ell} &= A'_{ij} B_k^{\cdot\ell} \\ &= L_{i'}^m L_{j'}^n A_{mn} L_{k'}^p L_q^{\ell'} B_p^{\cdot q} \\ &= L_{i'}^m L_{j'}^n L_{k'}^p L_q^{\ell'} A_{mn} B_p^{\cdot q} \\ &= L_{i'}^m L_{j'}^n L_{k'}^p L_q^{\ell'} C_{mnp}{}^{\cdot q},\end{aligned}$$

which follows the transformation law of a rank-four mixed tensor (with three covariant and one contravariant components).

# OUTER PRODUCT AND DIFFERENT STRUCTURES - CONCLUSION

## Conclusion

Using similar arguments as in the previous example, we can show that the outer product of any two tensors is a tensor.

Moreover, the structure of the new tensor is given by a concatenation of the structures of the tensors in the outer product.

## Examples

- The outer product between the mixed tensors  $A_i^j$  and  $B_k^\ell$  is the mixed tensor

$$C_{i \cdot k}^{\cdot j \cdot \ell} = A_i^j B_k^\ell.$$

- The outer product between the mixed tensors  $D_{ij}^{\cdot \cdot k}$  and  $E_{\cdot m}^{\ell \cdot n}$  is the mixed tensor

$$F_{ij \cdot \cdot m}^{\cdot \cdot k \ell \cdot n} = D_{ij}^{\cdot \cdot k} E_{\cdot m}^{\ell \cdot n}.$$

## Outer product and different ranks

We have seen that tensors of **different ranks** cannot be summed. But can we take the **outer product of tensors with different ranks**?

## Answer

Yes! The resulting tensor will have a rank equal to the sum of the ranks of the tensors involved in the outer product.

## Example

Consider the tensor  $C_{ijk..}{}^{\ell m} = A_{ijk}B^{\ell m}$ .

This is a rank-five tensor, resulting from the outer product of a rank-three tensor  $A_{ijk}$  and a rank-two tensor  $B^{\ell m}$ .

**Task:** Check this is in fact a rank-five tensor.

# OUTER PRODUCT IS NON-COMMUTATIVE

Outer product is non-commutative

Tensor multiplication is **non-commutative**.

Example of non-commutativity in rank two

Let us show that the outer product between two rank-two tensors  $A_{ik}$  and  $B_{\ell m}$  is non-commutative. That is,  $A_{ik}B_{\ell m} \neq B_{\ell m}A_{ik}$ .

For instance, assume

$$A_{ik} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 1 & 3 & 4 \end{pmatrix} \quad \text{and} \quad B_{\ell m} = \begin{pmatrix} 1 & 1 & 7 \\ 6 & 7 & 4 \\ 1 & 6 & 3 \end{pmatrix}.$$

Then

$$A_{12}B_{32} = 1 \cdot 6 = 6, \quad \text{whereas}$$

$$B_{12}A_{32} = 1 \cdot 3 = 3.$$

## Outer product of multiple tensors

Multiplication of any number of tensors of arbitrary rank is defined similarly.

For instance,

$$P_{ijklmn} = A_{ij}B_{kl}C_{mn}$$
$$C_{\cdot kl}^{i \cdot \cdot mnp} = A_{\cdot kl}^i B^{mn} C^p.$$

The **rank of a tensor product** is the sum of the rank of the factors.

In the examples above,  $P_{ijklmn}$  and  $C_{\cdot kl}^{i \cdot \cdot mnp}$  are both rank-six tensors.

# OUTER PRODUCT - NOTATION

## Outer product

Multiplication of tensors is called the **(outer) tensor product**.

The symbol  $\otimes$  is used to denote it:

$$\mathbf{C} = \mathbf{A} \otimes \mathbf{B}.$$

## Example of outer product

In our first example today, the tensor we called  $C_{iklm} = A_{ik}B_{lm}$  is just the outer product

$$\mathbf{A} \otimes \mathbf{B} = A_{ik}B_{lm}$$

between the tensors  $\mathbf{A} = [A_{ik}]$  and  $\mathbf{B} = [B_{ik}]$ .

# OUTER PRODUCT VISUALISATION - EXAMPLE 1

## Example 1

Let us visualise the outer products between

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = (1 \ 7).$$

We have

$$\mathbf{v}_1 \otimes \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \otimes \mathbf{v}_2 = \begin{pmatrix} 2 \cdot \mathbf{v}_2 \\ 0 \cdot \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 2(1 \ 7) \\ 0(1 \ 7) \end{pmatrix} = \begin{pmatrix} 2 & 14 \\ 0 & 0 \end{pmatrix}.$$

$$\begin{aligned} \mathbf{v}_2 \otimes \mathbf{v}_1 &= (1 \ 7) \otimes \mathbf{v}_1 = (1 \cdot \mathbf{v}_1 \quad 7 \cdot \mathbf{v}_1) = \left( 1 \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad 7 \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 2 & 14 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

## EXAMPLE 2

### Example 2

What is the (outer) tensor product of two matrices

$$\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}?$$

Note that these are rank two tensors, so the outer product must be a rank four tensor.

## EXAMPLE 2

### Example 2

We have

$$\begin{aligned}\mathbf{B} \otimes \mathbf{A} &= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \otimes \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= \begin{pmatrix} b_{11} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & b_{12} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ b_{21} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} & b_{22} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} b_{11}a_{11} & b_{11}a_{12} & b_{12}a_{11} & b_{12}a_{12} \\ b_{11}a_{21} & b_{11}a_{22} & b_{12}a_{21} & b_{12}a_{22} \\ b_{21}a_{11} & b_{21}a_{12} & b_{22}a_{11} & b_{22}a_{12} \\ b_{21}a_{21} & b_{21}a_{22} & b_{22}a_{21} & b_{22}a_{22} \end{pmatrix}\end{aligned}$$

# CONTRACTION OF TENSORS

## Definition

The operation of summing a tensor of rank  $n$  ( $n \geq 2$ ) over two of its indices is called a **contraction**.

## Example

A third-rank tensor  $P_{ikl}$  can be contracted in three different ways:

- $P_{iil} = P_{11l} + P_{22l} + P_{33l}, \quad (\ell = 1, 2, 3)$
- $P_{ikk} = P_{i11} + P_{i22} + P_{i33}, \quad (i = 1, 2, 3)$
- $P_{iki} = P_{1k1} + P_{2k2} + P_{3k3}, \quad (k = 1, 2, 3)$

## EXAMPLE - PART 2

### Example

Each such contraction of  $P_{ikl}$  is a first-rank tensor.

Assume we are working in Cartesian coordinates. Then  $P_{ikl}$  transforms according to the formula

$$P'_{ikl} = L_{im}L_{kn}L_{lr}P_{mnr}.$$

Hence

$$\begin{aligned}P'_{iil} &= L_{im}L_{in}L_{lr}P_{mnr} = \delta_{mn}L_{lr}P_{mnr} = L_{lr}P_{mnr}. \\P'_{ikk} &= L_{im}L_{kn}L_{kr}P_{mnr} = L_{im}\delta_{nr}P_{mnr} = L_{im}P_{mnn}. \\P'_{iki} &= L_{im}L_{kn}L_{ir}P_{mnr} = L_{kn}\delta_{mr}P_{mnr} = L_{kn}P_{mnm}.\end{aligned}$$

Thus, these quantities transform like a vector, as asserted.

## Repeated contraction

- The contraction of a tensor of rank  $n$  ( $n \geq 2$ ) produces a tensor of rank  $n - 2$ .
- This tensor of rank  $n - 2$  can then be contracted again (provided  $n \geq 4$ ), giving a tensor of rank  $n - 4$ .
- We can repeat this process many times, as long as we have high enough rank.
- Repeated contraction of a tensor of rank  $n$  eventually gives a scalar if  $n$  is even and a vector if  $n$  is odd.

Your turn!

Contract the second rank tensor

$$[A_{ij}] = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}.$$

## Definition

Contracting two tensors together is known as the **inner product**.

## Examples

The following expressions are inner products:

$$A_{ik}B_k, \quad \lambda_{iklm}B_{lm}, \quad A_iB_i.$$

## Example

We have previously seen lots of **inner products** involving the Kronecker delta  $\delta_{ij}$  and alternating tensor  $\epsilon_{ijk}$ .

For example,

$$\delta_{ij}\epsilon_{ijk} = \mathbf{0}_k,$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl},$$

$$\epsilon_{ijk}\epsilon_{ijm} = 2\delta_{km},$$

$$\epsilon_{ijk}\epsilon_{ijk} = 6.$$

# TASK

Your turn!

Given that

$$[R_{ik}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 7 & 2 \end{pmatrix}, \quad \mathbf{A} = \mathbf{i}_1 + \mathbf{i}_2 - \mathbf{i}_3.$$

find the inner products  $R_{ik}A_i$  and  $R_{ik}A_k$ .

Your turn!

Consider the following second-order tensors:

$$P = [P_{ik}] = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 1 & 4 \end{pmatrix}, \quad Q = [Q_{ik}] = \begin{pmatrix} 1 & 0 & 1 \\ 4 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Compute the contraction  $P_{jk}Q_{kj}$ .

## Contraction in generalised coordinates

- When considering generalised coordinate systems, contraction can be performed only on pairs of indices in **different positions**.
- In other words, one contracted index must be **covariant** and the other **contravariant**.
- Otherwise, the result of contraction will **not** be a tensor.

# EXAMPLE OF CONTRACTION IN GENERALISED COORDINATES

## Example

If we contract  $A_i^{k\ell}$  in the indices  $i$  and  $k$ , then  $A_i^{i\ell}$  is a tensor:

$$A_i^{i\ell} = L_i^m L_n^{i'} L_r^{\ell'} A_m^{nr} = \delta_n^m L_r^{\ell'} A_m^{nr} = L_r^{\ell'} A_n^{nr}.$$

However, contracting  $A_i^{k\ell}$  in the indices  $k$  and  $\ell$  gives a quantity whose transformation law is not that of a vector:

$$A_i^{kk} = L_i^m L_n^{k'} L_r^{k'} A_m^{nr}.$$

Similarly, in forming inner products, we can only sum over indices in **different positions**. For instance,

$$A^i B_i, \quad A_{ik} B^k, \quad \lambda_{ik}^{\cdot\cdot\ell m} B_{\ell m}, \quad \lambda_{ik \cdot m}^{\cdot\cdot\ell} B_{\ell}^{\cdot m}.$$

## Next time...

- Chapter 7: Tensor Fields
  - ▶ Preliminary,
  - ▶ Covariant differentiation.