TENSOR ANALYSIS

SLIDES WEEK 26 – LECTURE 1

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TENSOR FIELDS

Chapter 7: Tensor Fields

1. Preliminary,
2. Covariant differentiation,
3. Christoffel symbols,
4. Covariant differentiation of tensors,
5. Ricci's theorem,
6. Riemann-Christoffel tensor,
7. Ricci tensor.

PRELIMINARY

SO FAR

So far

So far, most bases we considered were **independent** from the position.

For instance, we can take the canonical basis of $\mathbb{R}^3$

$$\mathbf{e}_1 = (1, 0, 0), \qquad \mathbf{e}_2 = (0, 1, 0), \qquad \mathbf{e}_3 = (0, 0, 1).$$

We see that this is a fixed basis and does not depend on any parameter.

VECTOR FIELDS - PRELIMINARY

Definition

A **vector field** assigns a vector to every point in a subset of space.

Examples

- **Velocity Field:** Represents the velocity of a fluid at different points in space.
- **Gravitational Field:** Represents the gravitational force experienced by a mass at different points in space.
- **Local bases:** We could also define a basis $\mathbf{e}_1(\mathbf{x})$, $\mathbf{e}_2(\mathbf{x})$, $\mathbf{e}_3(\mathbf{x})$ that depends on the position.

DERIVATIVES OF TENSOR FIELDS

Derivatives of tensor fields

We have seen derivatives of tensor fields of rank zero. For instance,

- Gradient: $\nabla f = \nabla_i f$ or $\nabla_i u_j = \frac{\partial u_j}{\partial x_i}$,
- Divergence: $\nabla \cdot \mathbf{u} = \nabla_i u_i$,
- Curl: $\nabla \times \mathbf{u} = \epsilon_{ijk} \nabla_j u_k$.

Remark.

We can also take the derivative of rank 2 tensors such as

$$\frac{\partial \sigma_{jk}}{\partial x_i}.$$

This is the derivative of the conductivity tensor, which itself is a rank 3 tensor.

COVARIANT DIFFERENTIATION

EXPLANATION OF COVARIANT DIFFERENTIATION

Definition

Covariant differentiation is a way to differentiate vectors and tensors in curved space, ensuring the result is still a tensor.

Idea

- Imagine moving a vector along a curved surface.
- Covariant differentiation adjusts the vector to account for the curvature.

COVARIANT DIFFERENTIATION: CARTESIAN COORDINATES

Covariant differentiation: Cartesian Coordinates

Consider a **Cartesian** coordinate system with

- an orthonormal basis $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$, and
- a vector field $\mathbf{A} = \mathbf{A}(\mathbf{r})$ (i.e. a vector that varies in space).
- Assume the components of $\mathbf{A}$ are $A_1(\mathbf{r})$, $A_2(\mathbf{r})$, $A_3(\mathbf{r})$, that is

$$\mathbf{A} = A_1(\mathbf{r})\,\mathbf{i}_1 + A_2(\mathbf{r})\,\mathbf{i}_2 + A_3(\mathbf{r})\,\mathbf{i}_3 = A_j(\mathbf{r})\,\mathbf{i}_j .$$

Then the **differential** of $\mathbf{A}$ is

$$d\mathbf{A} = \underbrace{d(A_j(\mathbf{r})\,\mathbf{i}_j)}_{\text{product rule}} = \mathbf{i}_j\,dA_j(\mathbf{r}) + A_j(\mathbf{r})d\mathbf{i}_j .$$

COVARIANT DIFFERENTIATION: CARTESIAN COORDINATES - PART 2

Covariant differentiation: Cartesian Coordinates

Again,
$$d\mathbf{A} = \underbrace{d(A_j(\mathbf{r})\,\mathbf{i}_j)}_{\text{product rule}} = \mathbf{i}_j\,dA_j(\mathbf{r}) + A_j(\mathbf{r})d\mathbf{i}_j.$$

Since the basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ does not vary from point to point

$$d\mathbf{i}_j = 0, \qquad \forall j.$$

Then the **differential** of $\mathbf{A}$ is

$$\begin{aligned} d\mathbf{A} &= \mathbf{i}_j\,dA_j(\mathbf{r}) + \underbrace{A_j(\mathbf{r})d\mathbf{i}_j}_{=0} \\ &= \mathbf{i}_j\,dA_j(\mathbf{r}) \\ &= dA_1(\mathbf{r})\,\mathbf{i}_1 + dA_2(\mathbf{r})\,\mathbf{i}_2 + dA_3(\mathbf{r})\,\mathbf{i}_3 \\ &= (dA_1(\mathbf{r}),\, dA_2(\mathbf{r}),\, dA_3(\mathbf{r})). \end{aligned}$$

GENERAL COORDINATES WITH FIXED BASIS

General Coordinates with fixed basis

Consider a **generalised coordinate system** with a **fixed** basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ (constants).

Let $\mathbf{A}$ be a vector. We know it can be expanded via

$$\begin{aligned}\mathbf{A} &= \mathbf{e}^1 A_1(\mathbf{r}) + \mathbf{e}^2 A_2(\mathbf{r}) + \mathbf{e}^3 A_3(\mathbf{r})\\ &= \mathbf{e}_1 A^1(\mathbf{r}) + \mathbf{e}_2 A^2(\mathbf{r}) + \mathbf{e}_3 A^3(\mathbf{r}),\end{aligned}$$

where $A_1(\mathbf{r})$, $A_2(\mathbf{r})$, $A_3(\mathbf{r})$ are the **covariant components** and $A^1(\mathbf{r})$, $A^2(\mathbf{r})$, $A^3(\mathbf{r})$ are the contravariant components of $\mathbf{A} = \mathbf{A}(\mathbf{r})$.

Then, the **differential** of $\mathbf{A}$ is

$$\begin{aligned}d\mathbf{A} &= d(A_j(\mathbf{r})\mathbf{e}^j) = \mathbf{e}^j\, dA_j(\mathbf{r}),\\ d\mathbf{A} &= d(A^j(\mathbf{r})\mathbf{e}_j) = \mathbf{e}_j\, dA^j(\mathbf{r}),\end{aligned}$$

where $d\mathbf{e}_j = d\mathbf{e}^j = 0$.

GENERALISED COORDINATES WITH LOCAL BASIS

Generalised Coordinates with local basis

Now, suppose we have a **generalised coordinate system** with a **local basis** $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. That is, the basis varies from point to point:

$$\mathbf{e}_j = \mathbf{e}_j(x^1, x^2, x^3), \qquad \mathbf{e}^j = \mathbf{e}^j(x^1, x^2, x^3).$$

Because $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is a basis, any vector $\mathbf{A} = \mathbf{A}(\mathbf{r})$ can be expanded as

$$\mathbf{A} = \mathbf{e}^1 A_1 + \mathbf{e}^2 A_2 + \mathbf{e}^3 A_3 = \mathbf{e}_1 A^1 + \mathbf{e}_2 A^2 + \mathbf{e}_3 A^3.$$

However, it is **not true** that $d\mathbf{e}_j = d\mathbf{e}^j = 0$ here.

So the **differential** of $\mathbf{A}$ is

$$d\mathbf{A} = d(A_j \mathbf{e}^j) = \mathbf{e}^j \, dA_j + A_j \, d\mathbf{e}^j,$$
$$d\mathbf{A} = d(A^j \mathbf{e}_j) = \mathbf{e}_j \, dA^j + A^j \, d\mathbf{e}_j.$$

VECTOR DIFFERENTIAL USING LOCAL BASIS

Vector differential using local basis

In a generalised coordinate system with a **local basis** we have

$$d\mathbf{A} = \mathbf{e}^j \, dA_j + A_j \, d\mathbf{e}^j,$$
$$d\mathbf{A} = \mathbf{e}_j \, dA^j + A^j \, d\mathbf{e}_j.$$

We can also write

$$d\mathbf{A} = \frac{\partial \mathbf{A}}{\partial x^k} \, dx^k.$$

We then conclude that the **partial differentiation** is given by

$$\begin{aligned} \frac{\partial \mathbf{A}}{\partial x^k} &= \frac{\partial (A_j \mathbf{e}^j)}{\partial x^k} = \frac{\partial A_j}{\partial x^k} \, \mathbf{e}^j + A_j \frac{\partial \mathbf{e}^j}{\partial x^k} \\ &= \frac{\partial (A^j \mathbf{e}_j)}{\partial x^k} = \frac{\partial A^j}{\partial x^k} \, \mathbf{e}_j + A^j \frac{\partial \mathbf{e}_j}{\partial x^k}. \end{aligned}$$

EXAMPLE OF COVARIANT DIFFERENTIATION

Example

Consider a generalised coordinate system with local basis

$$\begin{aligned}\mathbf{e}^1 &= \mathbf{e}^1(x^1, x^2, x^3) = 2x^1\mathbf{i} + x^1\mathbf{j} + x^3\mathbf{k},\\ \mathbf{e}^2 &= \mathbf{e}^2(x^1, x^2, x^3) = x^2\mathbf{j} + x^2\mathbf{k},\\ \mathbf{e}^3 &= \mathbf{e}^3(x^1, x^2, x^3) = x^3\mathbf{k}.\end{aligned}$$

Find the partial derivative $\frac{\partial \mathbf{A}}{\partial x^1}$ of the vector

$$\mathbf{A} = x^2\mathbf{i} + 2x^1\mathbf{j} + x^1x^3\mathbf{k}.$$

SOLUTION

Solution

$$\frac{\partial \mathbf{A}}{\partial x^1} = \boxed{\frac{\partial A_j}{\partial x^1}\,\mathbf{e}^j} + A_j \frac{\partial \mathbf{e}^j}{\partial x^1}.$$

Since $\mathbf{A} = x^2\mathbf{i} + 2x^1\mathbf{j} + x^1x^3\mathbf{k}$, we have

$$\begin{aligned}\frac{\partial A_j}{\partial x^1}\,\mathbf{e}^j &= \frac{\partial A_1}{\partial x^1}\,\mathbf{e}^1 + \frac{\partial A_2}{\partial x^1}\,\mathbf{e}^2 + \frac{\partial A_3}{\partial x^1}\,\mathbf{e}^3 \\ &= \frac{\partial x^2}{\partial x^1}\,\mathbf{e}^1 + \frac{\partial 2x^1}{\partial x^1}\,\mathbf{e}^2 + \frac{\partial x^1x^3}{\partial x^1}\,\mathbf{e}^3 \\ &= 2\,\mathbf{e}^2 + x^3\,\mathbf{e}^3.\end{aligned}$$

Substituting $\mathbf{e}^2 = x^2\mathbf{j} + x^2\mathbf{k}$ and $\mathbf{e}^3 = x^3\mathbf{k}$, we obtain

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial x^1} &= 2(x^2\mathbf{j} + x^2\mathbf{k}) + x^3(x^3\mathbf{k}) \\ &= 2x^2\mathbf{j} + (2x^2 + (x^3)^2)\mathbf{k}.\end{aligned}$$

SOLUTION - PART 2

Solution

$$\frac{\partial \mathbf{A}}{\partial x^1} = \frac{\partial A_j}{\partial x^1}\,\mathbf{e}^j + \boxed{A_j \frac{\partial \mathbf{e}^j}{\partial x^1}}.$$

The second term is

$$\begin{aligned} A_j \frac{\partial \mathbf{e}^j}{\partial x^1} &= A_1 \frac{\partial \mathbf{e}^1}{\partial x^1} + A_2 \frac{\partial \mathbf{e}^2}{\partial x^1} + A_3 \frac{\partial \mathbf{e}^3}{\partial x^1} \\ &= A_1 \frac{\partial (2x^1\mathbf{i} + x^1\mathbf{j} + x^3\mathbf{k})}{\partial x^1} + A_2 \frac{\partial (x^2\mathbf{j} + x^2\mathbf{k})}{\partial x^1} + A_3 \frac{\partial (x^3\mathbf{k})}{\partial x^1} \\ &= A_1(2\mathbf{i} + \mathbf{j}) + 0 + 0 \\ &= 2A_1\mathbf{i} + A_1\mathbf{j}. \end{aligned}$$

SOLUTION - PART 3

Solution

We want to compute

$$\frac{\partial \mathbf{A}}{\partial x^1} = \frac{\partial A_j}{\partial x^1}\,\mathbf{e}^j + A_j\frac{\partial \mathbf{e}^j}{\partial x^1}.$$

We found

$$\begin{aligned} \frac{\partial A_j}{\partial x^1}\,\mathbf{e}^j &= 2x^2\mathbf{j} + (2x^2 + (x^3)^2)\mathbf{k}, \\ A_j\frac{\partial \mathbf{e}^j}{\partial x^1} &= 2A_1\mathbf{i} + A_1\mathbf{j}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial \mathbf{A}}{\partial x^1} &= 2x^2\mathbf{j} + (2x^2 + (x^3)^2)\mathbf{k} + 2A_1\mathbf{i} + A_1\mathbf{j} \\ &= 2A_1\mathbf{i} + (2x^2 + A_1)\mathbf{j} + (2x^2 + (x^3)^2)\mathbf{k}. \end{aligned}$$

PARTIAL DERIVATIVES ARE VECTORS

Partial derivatives are vectors

One can show that each partial derivative $\frac{\partial \mathbf{A}}{\partial x^k}$ $(k = 1, 2, 3)$ is a **vector**. To ease notation, let us denote

$$\mathbf{v}_k = \frac{\partial \mathbf{A}}{\partial x^k}, \text{ for } k = 1, 2, 3.$$

As any vector, the vector $\mathbf{v}_k$ also has covariant and contravariant components:

- **Contravariant:** $\mathbf{v}_k = (A_k)^1 \mathbf{e}_1 + (A_k)^2 \mathbf{e}_2 + (A_k)^3 \mathbf{e}_3 = (A_k)^i \mathbf{e}_i$.
- **Covariant:** $\mathbf{v}_k = (A_k)_1 \mathbf{e}^1 + (A_k)_2 \mathbf{e}^2 + (A_k)_3 \mathbf{e}^3 = (A_k)_i \mathbf{e}^i$.

For simplicity, we will write

- $A^i_{\cdot k} = (A_k)^i$.
- $A_{i,k} = (A_k)_i$.

Partial derivatives are vectors

Partial derivatives are vectors

These components $A^i_{\cdot k}$ and $A_{i,k}$ are themselves components of a second rank tensor called the **covariant derivative**.

Let us find a formula for $A_{i,k}$.

$$\frac{\partial \mathbf{A}}{\partial x^k} = \mathbf{v}_k = A_{1,k}\mathbf{e}^1 + A_{2,k}\mathbf{e}^2 + A_{3,k}\mathbf{e}^3.$$

Thus,

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial x^k} \cdot \mathbf{e}_i = \mathbf{v}_k \cdot \mathbf{e}_i &= \left(A_{1,k}\mathbf{e}^1 + A_{2,k}\mathbf{e}^2 + A_{3,k}\mathbf{e}^3\right) \cdot \mathbf{e}_i \\ &= A_{1,k}(\mathbf{e}^1 \cdot \mathbf{e}_i) + A_{2,k}(\mathbf{e}^2 \cdot \mathbf{e}_i) + A_{3,k}(\mathbf{e}^3 \cdot \mathbf{e}_i) \\ &= A_{i,k}.\end{aligned}$$

PARTIAL DERIVATIVES ARE VECTORS

Partial derivatives are vectors

We have shown

$$A_{i,k} = \frac{\partial \mathbf{A}}{\partial x^k} \cdot \mathbf{e}_i .$$

Similarly, we can show

$$A^{i}_{\cdot k} = \frac{\partial \mathbf{A}}{\partial x^k} \cdot \mathbf{e}^i .$$

SUMMARY: COVARIANT DIFFERENTIATION

Summary: Covariant differentiation

To summarise covariant differentiation:

$$\frac{\partial \mathbf{A}}{\partial x^k} = \frac{\partial A_j}{\partial x^k}\,\mathbf{e}^j + A_j\,\frac{\partial \mathbf{e}^j}{\partial x^k} = \frac{\partial A^j}{\partial x^k}\,\mathbf{e}_j + A^j\,\frac{\partial \mathbf{e}_j}{\partial x^k}.$$

$$A_{i,k} := \frac{\partial \mathbf{A}}{\partial x^k}\cdot\mathbf{e}_i \qquad \text{and} \qquad A^i_{,k} := \frac{\partial \mathbf{A}}{\partial x^k}\cdot\mathbf{e}^i.$$

Putting this together we have

$$A_{i,k} = \frac{\partial \mathbf{A}}{\partial x^k}\cdot\mathbf{e}_i = \frac{\partial A_i}{\partial x^k} + A_j\,\frac{\partial \mathbf{e}^j}{\partial x^k}\cdot\mathbf{e}_i,$$

$$A^i_{,k} = \frac{\partial \mathbf{A}}{\partial x^k}\cdot\mathbf{e}^i = \frac{\partial A^i}{\partial x^k} + A^j\,\frac{\partial \mathbf{e}_j}{\partial x^k}\cdot\mathbf{e}^i.$$

Christoffel Symbols

NEW NOTATION

New notation

Again, the **covariant** derivatives are defined by

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} + A_j \frac{\partial \mathbf{e}^j}{\partial x^k} \cdot \mathbf{e}_i,$$
$$A^i_{,k} = \frac{\partial A^i}{\partial x^k} + A^j \frac{\partial \mathbf{e}_j}{\partial x^k} \cdot \mathbf{e}^i$$

We want to simplify these expressions.

We will introduce a new symbol for that, which is called **Chistoffel symbol**.

CHRISTOFFEL SYMBOL

Christoffel Symbol

Covariant derivatives are defined by

$$\begin{aligned} A_{i,k} &= \frac{\partial A_i}{\partial x^k} + A_j \frac{\partial \mathbf{e}^j}{\partial x^k} \cdot \mathbf{e}_i, \\ A^i_{,k} &= \frac{\partial A^i}{\partial x^k} + A^j \frac{\partial \mathbf{e}_j}{\partial x^k} \cdot \mathbf{e}^i. \end{aligned}$$

We then define

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} \quad \text{and} \quad \Gamma^j{}_{ik} = -\mathbf{e}_i \cdot \frac{\partial \mathbf{e}^j}{\partial x^k}.$$

These are the **Christoffel symbols of the second kind** (which have 27 components).

COVARIANT DIFFERENTIATION IN TERMS OF CHRISTOFFEL SYMBOLS

Covariant differentiation in terms of Christoffel symbols

If we re-write the covariant derivatives using Christoffel symbols, we get

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} - \Gamma^j{}_{ik} A_j,$$

$$A^i_{,k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i{}_{jk} A^j.$$

The **Christoffel symbol of the second kind** are

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} \qquad \text{and} \qquad \Gamma^j{}_{ik} = -\mathbf{e}_i \cdot \frac{\partial \mathbf{e}^j}{\partial x^k}.$$

BACK TO A FIXED BASIS

Back to a fixed basis

Once more, covariant derivatives are defined by

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} - \Gamma^j{}_{ik} A_j,$$
$$A^i{}_{,k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i{}_{jk} A^j.$$

We have seen that, if the **basis is fixed**, then the Christoffel symbols vanish:

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} = 0.$$

Then the covariant derivatives $A_{i,k}$ and $A^i{}_{,k}$ reduce simply to

$$\partial A_{i,k} = \frac{\partial A_i}{\partial x^k} \quad \text{and} \quad \partial A^i{}_{,k} = \frac{\partial A^i}{\partial x^k}.$$

${\Gamma^i}_{jk}$ AS EXPANSION COEFFICIENTS

Christoffel symbols as expansion coefficients

We will now show that ${\Gamma^i}_{jk}$ are the **expansion coefficients** of the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. That is

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = {\Gamma^1}_{jk}\mathbf{e}_1 + {\Gamma^2}_{jk}\mathbf{e}_2 + {\Gamma^3}_{jk}\mathbf{e}_3 = {\Gamma^i}_{jk}\mathbf{e}_i.$$

To show this, let the following be the expansion of this vector:

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = {A^i}_{jk}\mathbf{e}_i.$$

Let us show that ${A^i}_{jk} = {\Gamma^i}_{jk}$.

${\Gamma^i}_{jk}$ AS EXPANSION COEFFICIENTS- PART 2

Christoffel symbols as expansion coefficients

To find these coefficients, we take the dot product of both sides of the equation with the dual basis vector $\mathbf{e}^m$:

$$\mathbf{e}^m \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} = \mathbf{e}^m \cdot ({A^i}_{jk}\mathbf{e}_i)$$

Since $\mathbf{e}^m \cdot \mathbf{e}_i = \delta_i^m$, we get:

$$\mathbf{e}^m \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} = {A^i}_{jk}\delta_i^m = A^m_{jk}.$$

Therefore

$${A^i}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} \qquad \text{while} \qquad {\Gamma^i}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}$$

Consequently, ${A^i}_{jk} = {\Gamma^i}_{jk}$, as required.

CHRISTOFFEL SYMBOLS OF FIRST KIND

Christoffel symbols of first kind

We just showed that ${\Gamma^i}_{jk}$ are the **expansion coefficients** of the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. That is

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = {\Gamma^1}_{jk}\mathbf{e}_1 + {\Gamma^2}_{jk}\mathbf{e}_2 + {\Gamma^3}_{jk}\mathbf{e}_3 = {\Gamma^i}_{jk}\mathbf{e}_i.$$

We can also expand the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the **dual basis** $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$.

The new coefficients are denoted by Γ_{ijk}. That is

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = \Gamma_{ijk}\mathbf{e}^i \quad \Longrightarrow \quad \Gamma_{ijk} = \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

These new symbols are called the **Christoffel symbols of the first kind**.

DEFINITIONS: TWO TYPES OF CHRISTOFFEL SYMBOL

Definitions.

- Christoffel symbols of **second kind**:

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

- Christoffel symbols of **first kind**:

$$\Gamma_{ijk} = \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

CHRISTOFFEL SYMBOLS AND METRIC TENSORS

Christoffel symbols and metric tensors

The Christoffel symbols can be expressed in terms of the **metric tensors**.

- Christoffel symbols of second kind

$$\Gamma^{i}{}_{jk} = \mathbf{e}^{i} \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} \quad \Longrightarrow \quad \frac{\partial \mathbf{e}_j}{\partial x^k} = \Gamma^{i}{}_{jk} \mathbf{e}_i .$$

- Christoffel symbols of first kind

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = \Gamma_{ijk} \mathbf{e}^{i} \quad \Longrightarrow \quad \Gamma_{ijk} = \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} .$$

We can then write

$$\Gamma_{ijk} = \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x_k} = \mathbf{e}_i \cdot \Gamma^{\ell}{}_{jk} \mathbf{e}_\ell = (\mathbf{e}_i \cdot \mathbf{e}_\ell) \Gamma^{\ell}{}_{jk} = g_{i\ell} \Gamma^{\ell}{}_{jk} .$$

CHRISTOFFEL SYMBOLS AND METRIC TENSORS - PART 2

Christoffel symbols and metric tensors

Similarly, we can write

$$\Gamma^i_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x_k} = \mathbf{e}^i \cdot \Gamma_{\ell jk}\mathbf{e}^\ell = (\mathbf{e}^i \cdot \mathbf{e}^\ell)\Gamma_{\ell jk} = g^{i\ell}\Gamma_{\ell jk}.$$

Hence, Christoffel symbols of the first and second kind are related via the metric tensor:

$$\Gamma_{ijk} = g_{i\ell}\Gamma^{\ell}{}_{jk}, \qquad \Gamma^{i}{}_{jk} = g^{i\ell}\Gamma_{\ell kj}.$$

SUMMARY: CHRISTOFFEL SYMBOLS AND THE METRIC TENSOR

Summary: Christoffel Symbols and the metric tensor

- Christoffel symbols of **second kind**

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

- Christoffel symbols of **first kind**

$$\Gamma_{ijk} = \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

Christoffel symbols of the first and second kind are related via the metric tensor

$$\Gamma_{ijk} = g_{i\ell}\Gamma^\ell{}_{jk}, \qquad \Gamma^i{}_{jk} = g^{i\ell}\Gamma_{\ell kj}.$$

SYMMETRY OF CHRISTOFFEL SYMBOLS

Symmetry of Christoffel symbols

The Christoffel symbols ${\Gamma^i}_{jk}$ and Γ_{ijk} are symmetric in the indices j and k. That is

$$\Gamma_{ijk} = \Gamma_{ikj}, \qquad \text{and} \qquad {\Gamma^i}_{jk} = {\Gamma^i}_{kj}.$$

In fact,

$$\begin{aligned} \Gamma_{ijk} &= \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} = \frac{\partial \mathbf{r}}{\partial x^i} \cdot \frac{\partial}{\partial x^k} \frac{\partial \mathbf{r}}{\partial x^j} && \left(\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i} \right) \\ &= \frac{\partial \mathbf{r}}{\partial x^i} \cdot \underbrace{\frac{\partial}{\partial x^j} \frac{\partial \mathbf{r}}{\partial x^k}}_{\text{re-order}} \\ &= \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_k}{\partial x^j} = \Gamma_{ikj}. \end{aligned}$$

SYMMETRY OF CHRISTOFFEL SYMBOLS- PART 2

Symmetry of Christoffel symbols

Simirly, we can show

$$\Gamma^{i}{}_{jk} = \mathbf{e}^{i} \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}$$

is symmetric in the indices j and k. That is

$$\Gamma^{i}{}_{jk} = \Gamma^{i}{}_{kj}.$$

Christoffel symbols are not tensors

Christoffel symbols are not tensors

We now show that the Christoffel symbols are **not** tensors.

Let us check how they transform under rotation of a coordinate system. We have

$$\begin{aligned}
\Gamma'_{ijk} &= \mathbf{e}'_i \cdot \frac{\partial \mathbf{e}'_j}{\partial x'^k} \\
&= \left(L^{\ell}_{i'}\mathbf{e}_\ell\right) \cdot \frac{\partial}{\partial x'^k}(L^m_{j'}\mathbf{e}_m) && (\mathbf{e}_k = L^m_{k'}\mathbf{e}_m) \\
&= \left(L^{\ell}_{i'}\mathbf{e}_\ell\right) \cdot \left(\frac{\partial}{\partial x^n}(L^m_{j'}\mathbf{e}_m)\frac{\partial x^n}{\partial x'^k}\right) && \text{(Chain rule)} \\
&= \left(L^{\ell}_{i'}\mathbf{e}_\ell\right) \cdot \left(\frac{\partial}{\partial x^n}(L^m_{j'}\mathbf{e}_m)L^n_{k'}\right) && \left(\frac{\partial x^n}{\partial x'^k} = L^n_{k'}\right).
\end{aligned}$$

CHRISTOFFEL SYMBOLS ARE NOT TENSORS-PART 2

Christoffel symbols are not tensors

So far, we have

$$\begin{aligned}
\Gamma'_{ijk} &= \left(L^{\ell}_{i'}\mathbf{e}_{\ell}\right)\cdot\left(\frac{\partial}{\partial x^{n}}(L^{m}_{j'}\mathbf{e}_{m})L^{n}_{k'}\right)\\
&= \left(L^{\ell}_{i'}\mathbf{e}_{\ell}\right)\cdot\left(L^{m}_{j'}\frac{\partial \mathbf{e}_{m}}{\partial x^{n}}L^{n}_{k'}+\mathbf{e}_{m}\frac{\partial L^{m}_{j'}}{\partial x^{n}}L^{n}_{k'}\right)\\
&= L^{\ell}_{i'}L^{m}_{j'}L^{n}_{k'}\underbrace{\left(\mathbf{e}_{\ell}\cdot\frac{\partial \mathbf{e}_{m}}{\partial x^{n}}\right)}_{\Gamma_{\ell mn}}+L^{\ell}_{i'}L^{n}_{k'}\frac{\partial L^{m}_{j'}}{\partial x^{n}}\underbrace{(\mathbf{e}_{\ell}\cdot\mathbf{e}_{m})}_{g_{\ell m}}\\
&= L^{\ell}_{i'}L^{m}_{j'}L^{n}_{k'}\Gamma_{\ell mn}+L^{\ell}_{i'}L^{n}_{k'}\frac{\partial L^{m}_{j'}}{\partial x^{n}}g_{\ell m}.
\end{aligned}$$

The above is not any of the tensor transformation laws. Thus the Christoffel symbol of the first kind is **not** a tensor.

CHRISTOFFEL SYMBOLS ARE NOT TENSORS - PART 2

Christoffel symbols are not tensors

Similarly, we can show that the transformation law of the Christoffel symbol of the second kind is given by

$$\Gamma^{i}{}_{jk}{}' = L^{i'}_{\ell} L^{m}_{j'} L^{n}_{k'} \Gamma^{\ell}{}_{mn} + L^{i'}_{m} L^{n}_{k'} \frac{\partial L^{m}_{j'}}{\partial x^{n}},$$

and thus the Christoffell symbol of the second kind is also **not** a tensor.

NEXT LECTURE

Next time...

Chapter 7:

- Christoffel symbols,
- Covariant differentiation of tensors.