Tensor Analysis

Slides week 26 – Lecture 2

Paula Lins

University of Lincoln

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Chapter 7: Tensor Fields

Today: Chapter 7–Tensor Fields

1. Preliminary,
2. Covariant differentiation,
3. Christoffel symbols,
4. Covariant differentiation of tensors,
5. Ricci's theorem,
6. Riemann-Christoffel tensor,
7. Ricci tensor.

Reminder

Covariant derivatives

The **covariant** derivatives are defined by

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} + A_j \frac{\partial \mathbf{e}^j}{\partial x^k} \cdot \mathbf{e}_i,$$
$$A^i_{,k} = \frac{\partial A^i}{\partial x^k} + A^j \frac{\partial \mathbf{e}_j}{\partial x^k}$$

Christoffel Symbol

The **covariant** derivatives are defined by

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} + A_j \frac{\partial \mathbf{e}^j}{\partial x^k} \cdot \mathbf{e}_i,$$
$$A^i_{,k} = \frac{\partial A^i}{\partial x^k} + A^j \frac{\partial \mathbf{e}_j}{\partial x^k} \cdot \mathbf{e}^i.$$

We then define

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} \quad \text{and} \quad \Gamma^j{}_{ik} = -\mathbf{e}_i \cdot \frac{\partial \mathbf{e}^j}{\partial x^k},$$

called **Christoffel symbols of the second kind**.

Covariant differentiation in terms of Christoffel symbols

Covariant differentiation in terms of Christoffel symbols

If we re-write the covariant derivatives using Christoffel symbols, we get

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} - \Gamma^j{}_{ik} A_j,$$

$$A^i_{,k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i{}_{jk} A^j,$$

where

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} \qquad \text{and} \qquad \Gamma^j{}_{ik} = -\mathbf{e}_i \cdot \frac{\partial \mathbf{e}^j}{\partial x^k}.$$

Back to a fixed basis

Once more, covariant derivatives are defined by

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} - \Gamma^j{}_{ik} A_j,$$
$$A^i{}_{,k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i{}_{jk} A^j.$$

We have seen that, if the **basis is fixed**, then the Christoffel symbols vanish:

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} = 0.$$

Then the covariant derivatives $A_{i,k}$ and $A^i{}_{,k}$ reduce simply to

$$\partial A_{i,k} = \frac{A_i}{\partial x^k} \quad \text{and} \quad \partial A^i{}_{,k} = \frac{A^i}{\partial x^k}.$$

Christoffel symbols as expansion coefficients

Notice that

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} \quad \Longrightarrow \quad \frac{\partial \mathbf{e}_j}{\partial x^k} = \Gamma^i{}_{jk} \mathbf{e}_i,$$

thus $\Gamma^i{}_{jk}$ are the **expansion coefficients** of the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. That is, if we write

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = A^1 \mathbf{e}_1 + A^2 \mathbf{e}_2 + A^3 \mathbf{e}_3,$$

then the coefficients A^1, A^2, A^3 are given by

$$A_i = \Gamma^i{}_{jk}.$$

Christoffel symbols as expansion coefficients - Part 2

Christoffel symbols as expansion coefficients

We can also expand the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the dual basis $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$.

We denote the coefficients of this expansion by Γ_{ijk}. That is

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = \Gamma_{ijk}\mathbf{e}^i \quad \Longrightarrow \quad \Gamma_{ijk} = \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

These new symbols are called the **Christoffel symbols of the first kind**.

Definitions: Two types of Christoffel Symbol

Definitions.

- Christoffel symbols of **second kind**:

$$\Gamma^{i}{}_{jk} = \mathbf{e}^{i} \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

- Christoffel symbols of **first kind**:

$$\Gamma_{ijk} = \mathbf{e}_{i} \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

Christoffel symbols and metric tensors

Christoffel symbols of the first and second kind are related via the metric tensor:

$$\Gamma_{ijk} = g_{i\ell}\Gamma^{\ell}{}_{jk}, \qquad \Gamma^{i}{}_{jk} = g^{i\ell}\Gamma_{\ell kj},$$

where the Christoffel symbols are

$$\Gamma^{i}{}_{jk} = \mathbf{e}^{i} \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} \quad \text{and} \quad \Gamma_{ijk} = \mathbf{e}_{i} \cdot \frac{\partial \mathbf{e}_j}{\partial x^k},$$

and the metric tensor is

$$g_{i\ell} = \mathbf{e}_i \cdot \mathbf{e}_\ell \quad \text{and} \quad g^{i\ell} = \mathbf{e}^i \cdot \mathbf{e}^\ell.$$

Summary: Christoffel Symbols and the Metric Tensor

Summary: Christoffel Symbols and the Metric Tensor

- Christoffel symbols of **second kind**

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

- Christoffel symbols of **first kind**

$$\Gamma_{ijk} = \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

Christoffel symbols of the first and second kind are related via the metric tensor

$$\Gamma_{ijk} = g_{i\ell}\Gamma^\ell{}_{jk}, \qquad \Gamma^i{}_{jk} = g^{i\ell}\Gamma_{\ell kj}.$$

Christoffel Symbols formula

Generalised coordinate system - Christoffel symbol formula

Generalised coordinate system - Christoffel symbol formula

Because Γ_{ijk} is symmetric in j, k, we can write

$$\Gamma_{ijk} = \frac{1}{2}(\Gamma_{ijk} + \Gamma_{ikj}).$$

Using the definition

$$\Gamma_{ijk} = \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k},$$

we get

$$\frac{1}{2}(\Gamma_{ijk} + \Gamma_{ikj}) = \frac{1}{2}\left(\mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} + \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_k}{\partial x^j}\right).$$

Generalised coordinate system - Christoffel symbol formula -Part 2

Generalised coordinate system - Christoffel symbol formula

We then obtain

$$
\begin{aligned}
\Gamma_{ijk} &= \frac{1}{2}\left(\mathbf{e}_i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} + \mathbf{e}_i \cdot \frac{\partial \mathbf{e}_k}{\partial x^j}\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x^k}(\mathbf{e}_i \cdot \mathbf{e}_j) + \frac{\partial}{\partial x^j}(\mathbf{e}_i \cdot \mathbf{e}_k) - \mathbf{e}_j \cdot \frac{\partial \mathbf{e}_i}{\partial x^k} - \mathbf{e}_k \cdot \frac{\partial \mathbf{e}_i}{\partial x^j}\right) \\
&= \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \mathbf{e}_j \cdot \frac{\partial \mathbf{e}_k}{\partial x^i} - \mathbf{e}_k \cdot \frac{\partial \mathbf{e}_j}{\partial x^i}\right) \\
&= \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial}{\partial x^i}(\mathbf{e}_j \cdot \mathbf{e}_k)\right), \\
&= \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right).
\end{aligned}
$$

Summary - Christoffel symbol formula

Christoffel symbol of the **first kind** is related to the **metric tensor** of the underlying generalised coodinate system via

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right).$$

Generalised coordinate system - Christoffel symbol formula - Second kind

Generalised coordinate system - Christoffel symbol formula

Since the Christoffel symbol of the **first kind** is given by

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right),$$

and we have the relation

$$\Gamma^i{}_{jk} = g^{i\ell}\Gamma_{\ell kj},$$

we can derive a formula for Christoffel symbols of the second kind:

$$\Gamma^i{}_{jk} = \frac{1}{2}g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right).$$

Example

Suppose a coordinate system $(x^1, x^2, x^3) = (\rho, \phi, \theta)$ has metric tensors given by

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = \sin\phi \sin\theta,$$

and the other components of the metric tensor are trivial. Using

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

find **all** Christoffel symbols of the first and second kind

EXAMPLE - PART 2

Example

In the coordinate system $(x^1, x^2, x^3) = (\rho, \phi, \theta)$, we have

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = \sin\phi \sin\theta,$$

and we know that the Christoffel symbols of **first kind** are given by

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right).$$

The first step is then to determine which partial derivatives $\frac{\partial g_{ij}}{\partial x^k}$ are non-zero.

Note that all g_{ij} are constant, except for $g_{22} = r^2$ and $g_{33} = \sin\phi \sin\theta$. Thus

$$\frac{\partial g_{ij}}{\partial x^k} = 0, \quad \text{unless} \quad (i, j, k) = (2, 2, 1), (3, 3, 2), (3, 3, 3).$$

Example

Since

$$\frac{\partial g_{ij}}{\partial x^k} = 0, \quad \text{unless} \quad (i,j,k) = (2,2,1), (3,3,2), (3,3,3),$$

and

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right),$$

the only **possibly** non-zero Christoffel symbols are the ones having

- 2, 2, 1, or
- 3, 3, 2, or
- 3, 3, 2

as indices (in some order).

Example

Since $(x^1, x^2, x^3) = (\rho, \phi, \theta)$ and

$$g_{22} = r^2, \quad g_{33} = \sin\phi \sin\theta,$$

we obtain

$$\frac{\partial g_{22}}{\partial x^1} = \frac{\partial r^2}{\partial r} = 2r$$

$$\frac{\partial g_{33}}{\partial x^2} = \frac{\partial(\sin\phi\sin\theta)}{\partial\phi} = \cos\phi\sin\theta,$$

$$\frac{\partial g_{33}}{\partial x^3} = \frac{\partial(\sin\phi\sin\theta)}{\partial\theta} = \sin\phi\cos\theta.$$

Example - Part 5

Example

Let us combine everything. We found

- The only posisbly non-zero Christoffel symvbols are

$$\Gamma_{122}, \quad \Gamma_{212}, \quad \Gamma_{221}, \quad \Gamma_{233}, \quad \Gamma_{323}, \quad \Gamma_{332}, \quad \Gamma_{333}.$$

- The non-zero partial derivatives are

$$\frac{\partial g_{22}}{\partial x^1} = 2r, \quad \frac{\partial g_{33}}{\partial x^2} = \cos\phi \sin\theta, \quad \frac{\partial g_{33}}{\partial x^3} \sin\phi \cos\theta.$$

- The Christoffel symbols of first kind are given by

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right).$$

Example - Part 6

Example

Thus

$$\Gamma_{122} = \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right)$$

$$= -\frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2}2r = -r.$$

$$\Gamma_{212} = \frac{1}{2}\left(\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^2}\right)$$

$$= \frac{1}{2}\frac{\partial g_{22}}{\partial x^1} = \frac{1}{2}2r = r.$$

Since the Chistoffel symbols are symmetric in their last two indices, we also conclude

$$\Gamma_{221} = \Gamma_{212} = r.$$

Example

Moreover

$$\begin{aligned}\Gamma_{233} &= \frac{1}{2}\left(\frac{\partial g_{23}}{\partial x^3} + \frac{\partial g_{23}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^2}\right) \\ &= -\frac{1}{2}\frac{\partial g_{33}}{\partial x^2} = -\frac{1}{2}(\cos\phi\sin\theta).\end{aligned}$$

$$\begin{aligned}\Gamma_{323} = \Gamma_{332} &= \frac{1}{2}\left(\frac{\partial g_{32}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^3} - \frac{\partial g_{23}}{\partial x^3}\right) \\ &= \frac{1}{2}\frac{\partial g_{33}}{\partial x^3} = \frac{1}{2}(\cos\phi\sin\theta),\end{aligned}$$

$$\begin{aligned}\Gamma_{333} &= \frac{1}{2}\left(\frac{\partial g_{33}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^3}\right) \\ &= \frac{1}{2}\frac{\partial g_{33}}{\partial x^3} = \frac{1}{2}(\sin\phi\cos\theta).\end{aligned}$$

Example - Part 8

Example

Thus, the Christoffel symbols of first kind are given by

$$\begin{aligned} \Gamma_{122} &= -r, & \Gamma_{221} &= \Gamma_{212} = r, \\ \Gamma_{233} &= -\tfrac{1}{2}(\cos\phi \sin\theta) & \Gamma_{323} &= \Gamma_{332} = \tfrac{1}{2}(\cos\phi \sin\theta), \\ \Gamma_{333} &= \tfrac{1}{2}(\sin\phi \cos\theta), \end{aligned}$$

and $\Gamma_{ijk} = 0$, for all other values of i, j, k.

Example - Part 9

Example

To find the Christoffel symbols of **second kind**, we use the formula

$$\Gamma^i{}_{jk} = g^{i\ell}\Gamma_{\ell kj} \quad \text{and } g^{ii} = 1/g_{ii}.$$

We have

$$\begin{aligned}\Gamma^1_{22} &= g^{1\ell}\Gamma_{\ell 22} = g^{11}\Gamma_{122} + g^{12}\Gamma_{222} + g^{13}\Gamma_{322} \\ &= g^{11}\Gamma_{122} = \frac{1}{g_1 1}\Gamma_{122} = \frac{1}{1}\cdot(-r) = -r.\end{aligned}$$

Similarly,

$$\Gamma^2_{21} = \Gamma^2_{12} = g^{2\ell}\Gamma_{\ell 21} = g^{22}\Gamma_{221} = \frac{1}{r^2}\cdot r = \frac{1}{r}.$$

Example - Last step

Example

Following the same steps for the other Christoffel symbols of **second kind**, we find

$$\Gamma^1_{22} = -r,$$

$$\Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{r},$$

$$\Gamma^2_{33} = -\frac{\cos\phi \sin\theta}{2r^2},$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \frac{1}{2\tan\phi},$$

$$\Gamma^3_{33} = \frac{1}{2\tan\theta},$$

and $\Gamma^i_{jk} = 0$, for all other values of i, j, k.

Your turn!

In Exercise 5.5, you showed that, in spherical coordinates $(x^1, x^2, x^3) = (r, \phi, \theta)$, the metric tensors are

$$g_{11} = 1, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2 \phi,$$

and the other components of the metric tensor are trivial. Using

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

find **all** Christoffel symbols of the first and second kind

Covariant differentiation of tensors

Recap of vectors

Recall that the **covariant derivatives** of a vector can be written in terms of the Christoffel symbol of the second kind:

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} - \Gamma^j{}_{ik} A_j,$$
$$A^i_{,k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i{}_{jk} A^j.$$

Second rank tensors

Likewise, we have formulae for the covariant derivative of components of second-rank tensors:

$$T_{ik,\ell} = \frac{\partial T_{ik}}{\partial x^\ell} - \Gamma^m_{i\ell} T_{mk} - \Gamma^m_{k\ell} T_{im}$$

$$T^{ik}_{,\ell} = \frac{\partial T^{ik}}{\partial x^\ell} + \Gamma^i_{\ m\ell} T^{mk} + \Gamma^k_{\ m\ell} T^{im}$$

$$T^i_{.k,\ell} = \frac{\partial T^i_{.k}}{\partial x^\ell} + \Gamma^i_{\ m\ell} T^m_{.k} - \Gamma^m_{k\ell} T^i_{.m}.$$

These quantities $T_{ik,\ell}, T^{ik}_{,\ell}, T^i_{.k,\ell}$ transform under the appropriate tensor transformation law respectively (Exercise).

Next Lecture

Next time...

Chapter 7:

- Ricci's theorem.