TENSOR ANALYSIS

SLIDES WEEK 30 – LECTURE 1

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CHAPTER 7: TENSOR FIELDS

Today: Chapter 7–Tensor Fields

1. Preliminary,
2. Covariant differentiation,
3. Christoffel symbols,
4. Covariant differentiation of tensors,
5. Ricci's theorem,
6. Riemann-Christoffel tensor,
7. Ricci tensor.

REMINDER

GENERALISED COORDINATES WITH LOCAL BASIS

Generalised Coordinates with local basis

- Recall: We are now considering generalised coordinate systems with **local bases**.

- A **local basis** is a basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ that varies from point to point:

$$\mathbf{e}_j = \mathbf{e}_j(x^1, x^2, x^3), \qquad \mathbf{e}^j = \mathbf{e}^j(x^1, x^2, x^3).$$

- Because $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is a basis, any vector $\mathbf{A} = \mathbf{A}(\mathbf{r})$ can be expanded as

$$\begin{aligned} \mathbf{A} &= \mathbf{e}^1 A_1 + \mathbf{e}^2 A_2 + \mathbf{e}^3 A_3 \text{ and} \\ &= \mathbf{e}_1 A^1 + \mathbf{e}_2 A^2 + \mathbf{e}_3 A^3. \end{aligned}$$

DIFFERENTIAL OF A VECTOR

Differential of a vector

- Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be a local basis.
- Let $\mathbf{A}$ be a vector. Then

$$\mathbf{A} = \mathbf{e}^1 A_1 + \mathbf{e}^2 A_2 + \mathbf{e}^3 A_3 = A_j \mathbf{e}^j$$

$$\text{and } \mathbf{A} = \mathbf{e}_1 A^1 + \mathbf{e}_2 A^2 + \mathbf{e}_3 A^3 = A^j \mathbf{e}_j.$$

- The **differential** of the vector $\mathbf{A}$ is then

$$d\mathbf{A} = d(A_j \mathbf{e}^j) = \mathbf{e}^j \, dA_j + A_j \, d\mathbf{e}^j,$$

$$d\mathbf{A} = d(A^j \mathbf{e}_j) = \mathbf{e}_j \, dA^j + A^j \, d\mathbf{e}_j.$$

- Since $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is a **local basis**, we **do not** necessarily have that $d\mathbf{e}^j$ and $d\mathbf{e}_j$ are zero.

PARTIAL DERIVATIVES ARE VECTORS

Partial derivatives are vectors

- Let $\mathbf{A}$ be a vector (i.e. a rank-one tensor).
- Then each partial derivative $\frac{\partial \mathbf{A}}{\partial x^k}$ $(k = 1, 2, 3)$ is a **vector**.
- As any vector, the vector $\frac{\partial \mathbf{A}}{\partial x^k}$ also has covariant and contravariant components:
  - **Contravariant:** $\mathbf{v}_k = (A_k)^1\mathbf{e}_1 + (A_k)^2\mathbf{e}_2 + (A_k)^3\mathbf{e}_3 = (A_k)^i\mathbf{e}_i$.
  - **Covariant:** $\mathbf{v}_k = (A_k)_1\mathbf{e}^1 + (A_k)_2\mathbf{e}^2 + (A_k)_3\mathbf{e}^3 = (A_k)_i\mathbf{e}^i$.

For simplicity, we write

- $A^i_{\cdot k} = (A_k)^i$.
- $A_{i,k} = (A_k)_i$.

PARTIAL DERIVATIVES ARE VECTORS

Partial derivatives are vectors

- We have computed the following formulae for the covariant components:

$$A_{i,k} = \frac{\partial \mathbf{A}}{\partial x^k} \cdot \mathbf{e}_i = \frac{\partial A_i}{\partial x^k} + A_j \frac{\partial \mathbf{e}^j}{\partial x^k} \cdot \mathbf{e}_i.$$

- Similarly, we showed that the contravariant components are given by

$$A^i_{\cdot k} = \frac{\partial \mathbf{A}}{\partial x^k} \cdot \mathbf{e}^i = \frac{\partial A^i}{\partial x^k} + A^j \frac{\partial \mathbf{e}_j}{\partial x^k} \cdot \mathbf{e}^i.$$

CHRISTOFFEL SYMBOL

Christoffel Symbol

Again, the covariant derivatives are given by

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} + A_j \frac{\partial \mathbf{e}^j}{\partial x^k} \cdot \mathbf{e}_i,$$

$$A^i_{,k} = \frac{\partial A^i}{\partial x^k} + A^j \frac{\partial \mathbf{e}_j}{\partial x^k} \cdot \mathbf{e}^i.$$

We then define

$$\Gamma^i{}_{jk} = \mathbf{e}^i \cdot \frac{\partial \mathbf{e}_j}{\partial x^k} \quad \text{and} \quad \Gamma^j{}_{ik} = -\mathbf{e}_i \cdot \frac{\partial \mathbf{e}^j}{\partial x^k}.$$

These are the **Christoffel symbols of the second kind**. Using such notation

$$A_{i,k} = \frac{\partial A_i}{\partial x^k} - \Gamma^j{}_{ik} A_j,$$

$$A^i_{,k} = \frac{\partial A^i}{\partial x^k} + \Gamma^i{}_{jk} A^j.$$

${\Gamma^i}_{jk}$ AS EXPANSION COEFFICIENTS

Christoffel symbols as expansion coefficients

- Recall that the partial derivatives $\frac{\partial \mathbf{A}}{\partial x^k}$ of any vector $\mathbf{A}$ are vectors.
- Thus, we can expand each $\frac{\partial \mathbf{A}}{\partial x^k}$ with respect to a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and with respect with its dual e^1, e^2, e^3.
- It turns out that the ${\Gamma^i}_{jk}$ are the **expansion coefficients** of the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = {\Gamma^1}_{jk}\mathbf{e}_1 + {\Gamma^2}_{jk}\mathbf{e}_2 + {\Gamma^3}_{jk}\mathbf{e}_3 = {\Gamma^i}_{jk}\mathbf{e}_i.$$

CHRISTOFFEL SYMBOLS OF FIRST KIND

Christoffel symbols of first kind

- The same way ${\Gamma^i}_{jk}$ are the **expansion coefficients** of the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = {\Gamma^1}_{jk}\mathbf{e}_1 + {\Gamma^2}_{jk}\mathbf{e}_2 + {\Gamma^3}_{jk}\mathbf{e}_3 = {\Gamma^i}_{jk}\mathbf{e}_i,$$

  we can also expand the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the **dual basis $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$**.

- The new coefficients are denoted by Γ_{ijk}. That is

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = \Gamma_{1jk}\mathbf{e}^1 + \Gamma_{2jk}\mathbf{e}^2 + \Gamma_{3jk}\mathbf{e}^3 = \Gamma_{ijk}\mathbf{e}^i.$$

- These new symbols are called the **Christoffel symbols of the first kind**.

SUMMARY: CHRISTOFFEL SYMBOLS AND THE METRIC TENSOR

Summary: Christoffel Symbols and the metric tensor

- Christoffel symbols of **second kind** are defined as

$$\Gamma^{i}{}_{jk} = \mathbf{e}^{i} \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

- Christoffel symbols of **first kind** are defined as

$$\Gamma_{ijk} = \mathbf{e}_{i} \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

Christoffel symbols of the first and second kind are related via the metric tensor

$$\Gamma_{ijk} = g_{i\ell}\Gamma^{\ell}{}_{jk}, \qquad \Gamma^{i}{}_{jk} = g^{i\ell}\Gamma_{\ell kj}.$$

PROPERTIES OF CHRISTOFFEL SYMBOLS

Properties of Christoffel symbols

- **Symmetry in the last two indices:** The Christoffel symbols ${\Gamma^i}_{jk}$ and Γ_{ijk} are symmetric in the indices j and k. That is

$$\Gamma_{ijk} = \Gamma_{ikj}, \qquad \text{and} \qquad {\Gamma^i}_{jk} = {\Gamma^i}_{kj}.$$

- They are **not tensors**!

- We have a formulae relating Christoffel symbols to the **metric tensor**:

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right),$$

$$\Gamma^i_{\cdot\, jk} = \frac{1}{2} g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right).$$

TASK

Your turn!

Consider coordinates $(x^1, x^2, x^3) = (u,\, v,\, w)$ with position vector given by

$$\mathbf{r} = e^u\, \mathbf{i}_1 + e^v\, \mathbf{i}_2 + w\, \mathbf{i}_3.$$

Using

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

find all Christoffel symbols of the first and second kind.

RICCI'S THEOREM

RICCI'S THEOREM:

Ricci's Theorem.

The covariant derivative of the metric tensor g_{ik} vanishes.

Proof.

We have the general formula

$$T_{ik,\ell} = \frac{\partial T_{ik}}{\partial x^\ell} - \Gamma^m_{i\ell} T_{mk} - \Gamma^m_{k\ell} T_{im}.$$

Applying it to $T_{ik,\ell} = g_{ik,\ell}$, we get

$$g_{ik,\ell} = \frac{\partial g_{ik}}{\partial x^\ell} - \Gamma^m_{i\ell} g_{mk} - \Gamma^m_{k\ell} g_{im}.$$

Proof of Ricci's Theorem

Proof.

Recall the relationship between first and second Christoffel symbols

$$\Gamma_{ijk} = g_{i\ell}\Gamma^{\ell}{}_{jk}.$$

Thus,

$$\begin{aligned} g_{ik,\ell} &= \frac{\partial g_{ik}}{\partial x^{\ell}} - \Gamma^{m}_{i\ell} g_{mk} - \Gamma^{m}_{k\ell} g_{im} \\ &= \frac{\partial g_{ik}}{\partial x^{\ell}} - \Gamma_{ik\ell} - \Gamma_{ki\ell}. \end{aligned}$$

Using the relation

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^{k}} + \frac{\partial g_{ik}}{\partial x^{j}} - \frac{\partial g_{jk}}{\partial x^{i}}\right)$$

we get

$$g_{ik,\ell} = \frac{\partial g_{ik}}{\partial x^{\ell}} - \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^{\ell}} + \frac{\partial g_{i\ell}}{\partial x^{k}} - \frac{\partial g_{k\ell}}{\partial x^{i}}\right) - \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^{\ell}} + \frac{\partial g_{k\ell}}{\partial x^{i}} - \frac{\partial g_{i\ell}}{\partial x^{k}}\right).$$

PROOF OF RICCI'S THEOREM - PART 2

Proof.

Now,

$$\begin{aligned} g_{ik,\ell} &= \frac{\partial g_{ik}}{\partial x^\ell} - \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^\ell} + \cancel{\frac{\partial g_{i\ell}}{\partial x^k}} - \cancel{\frac{\partial g_{k\ell}}{\partial x^i}}\right) - \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^\ell} + \cancel{\frac{\partial g_{k\ell}}{\partial x^i}} - \cancel{\frac{\partial g_{i\ell}}{\partial x^k}}\right) \\ &= \frac{\partial g_{ik}}{\partial x^\ell} - \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^\ell} + \frac{\partial g_{ik}}{\partial x^\ell}\right) \\ &= 0. \end{aligned}$$

□

USEFUL FORMULA

Useful formula

By Ricci's theorem, we have

$$\begin{aligned} g_{ik,\ell} &= \frac{\partial g_{ik}}{\partial x^{\ell}} - \Gamma^{m}_{i\ell} g_{mk} - \Gamma^{m}_{k\ell} g_{im} \\ &= \frac{\partial g_{ik}}{\partial x^{\ell}} - \Gamma_{ik\ell} - \Gamma_{ki\ell} = 0 \end{aligned}$$

which yields the following useful formula

$$\frac{\partial g_{ik}}{\partial x^{\ell}} = \Gamma_{ik\ell} + \Gamma_{ki\ell}.$$

In particular, in **orthogonal coordinates**

$$\Gamma_{ik\ell} + \Gamma_{ki\ell} = 0 \implies \Gamma_{ik\ell} = -\Gamma_{ki\ell}.$$

EXAMPLE - RICCI'S THEOREM

Example

Consider a **orthogonal** coordinate system $(x^1, x^2, x^3) = (\rho,\ \tau,\ \theta)$, the metric tensor g_{22} is

$$g_{22} = \rho^2 - \tau^2.$$

Find Γ_{122}, Γ_{212} and Γ_{221}.

Solution: We have

$$\begin{aligned}\Gamma_{122} &= \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right) = -\frac{1}{2}\ \frac{\partial g_{22}}{\partial x^1} \\ &= -\frac{1}{2}\ \frac{\partial}{\partial \rho}(\rho^2 - \tau^2) \\ &= -\frac{1}{2}(2\rho) = -\rho.\end{aligned}$$

EXAMPLE - RICCI'S THEOREM - PART 2

Example

We found

$$\Gamma_{122} = -\rho.$$

We still need to find Γ_{212} and Γ_{221}.

Solution: By Ricci's Theorem

$$\Gamma_{212} = -\Gamma_{122} = \rho.$$

By the symmetry of the Christoffel symbol in the last two components

$$\Gamma_{221} = \Gamma_{212} = \rho.$$

REMARK - RICCI'S THEOREM

Remark

From Ricci's Theorem, we derived that in **orthogonal coordinates**

$$\Gamma_{ik\ell} = -\Gamma_{ki\ell}.$$

However, this only works for Christoffel symbols of **first kind**!!

Example

Consider again the **orthogonal** coordinate system $(x^1, x^2, x^3) = (\rho,\ \tau,\ \theta)$, where the metric tensor g_{22} is

$$g_{22} = \rho^2 - \tau^2, \qquad \text{and we also assume} \qquad g_{11} = 1.$$

Let us find Γ^1_{22} and Γ^2_{12} to show that $\Gamma^1_{22} \neq -\Gamma^2_{12}$.

REMARK - RICCI'S THEOREM - PART 2

Example

Again

$$g_{22} = \rho^2 - \tau^2, \quad \text{and we also assume} \quad g_{11} = 1.$$

Since this system is orthogonal

$$g^{11} = \frac{1}{g_{11}} = 1, \qquad g^{22} = \frac{1}{g_{22}} = \frac{1}{\rho^2 - \tau^2}.$$

Using

$$\Gamma^i_{jk} = g^{i\ell}\Gamma_{\ell jk}, \qquad \Gamma_{122} = -\rho, \qquad \Gamma_{212} = \rho,$$

we find

$$\Gamma^1_{22} = g^{1\ell}\Gamma_{\ell 22} = g^{11}\Gamma_{122} = -\rho.$$

$$\Gamma^2_{12} = g^{2\ell}\Gamma_{\ell 12} = g^{22}\Gamma_{212} = \frac{1}{\rho^2 - \tau^2} \cdot \rho = \frac{\rho}{\rho^2 - \tau^2}.$$

REMARK - RICCI'S THEOREM - PART 3

Example

We found

$$\Gamma^1_{22} = -\rho.$$

$$\Gamma^2_{12} = \frac{\rho}{\rho^2 - \tau^2}.$$

So, $\Gamma^1_{22} \neq -\Gamma^2_{12}$, as claimed.

Remark

The symmetry on the last indices holds for Cristoffel symbols of second kind. Thus, it is true that

$$\Gamma^2_{21} = \Gamma^2_{12} = -\frac{\rho}{\rho^2 - \tau^2}.$$

NEXT LECTURE

Next time...

Chapter 7:

- Riemann-Christoffel tensor.