TENSOR ANALYSIS

SLIDES WEEK 30 – LECTURE 2

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CHAPTER 7: TENSOR FIELDS

Today: Chapter 7–Tensor Fields

1. Preliminary,
2. Covariant differentiation,
3. Christoffel symbols,
4. Covariant differentiation of tensors,
5. Ricci's theorem,
6. Riemann-Christoffel tensor,
7. Ricci tensor.

REMINDER

GENERALISED COORDINATES WITH LOCAL BASIS

Generalised Coordinates with local basis

- Recall: We are now considering generalised coordinate systems with **local bases**.
- A **local basis** is a basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ that varies from point to point:

$$\mathbf{e}_j = \mathbf{e}_j(x^1, x^2, x^3), \qquad \mathbf{e}^j = \mathbf{e}^j(x^1, x^2, x^3).$$

CHRISTOFFEL SYMBOLS AND THE METRIC TENSOR

Christoffel Symbols and the metric tensor

- Christoffel symbols of **second kind** are defined as

$$\Gamma^{i}{}_{jk} = \mathbf{e}^{i} \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

- Christoffel symbols of **first kind** are defined as

$$\Gamma_{ijk} = \mathbf{e}_{i} \cdot \frac{\partial \mathbf{e}_j}{\partial x^k}.$$

Christoffel symbols of the first and second kind are related via the metric tensor

$$\Gamma_{ijk} = g_{i\ell}\Gamma^{\ell}{}_{jk}, \qquad \Gamma^{i}{}_{jk} = g^{i\ell}\Gamma_{\ell kj}.$$

$\Gamma^i{}_{jk}$ AS EXPANSION COEFFICIENTS

Christoffel symbols as expansion coefficients

- Recall that the partial derivatives $\frac{\partial \mathbf{A}}{\partial x^k}$ of any vector $\mathbf{A}$ are vectors.
- Thus, we can expand each $\frac{\partial \mathbf{A}}{\partial x^k}$ with respect to a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and with respect with its dual e^1, e^2, e^3.
- It turns out that the $\Gamma^i{}_{jk}$ are the **expansion coefficients** of the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = \Gamma^1{}_{jk}\mathbf{e}_1 + \Gamma^2{}_{jk}\mathbf{e}_2 + \Gamma^3{}_{jk}\mathbf{e}_3 = \Gamma^i{}_{jk}\mathbf{e}_i.$$

CHRISTOFFEL SYMBOLS OF FIRST KIND

Christoffel symbols of first kind

- The same way ${\Gamma^i}_{jk}$ are the **expansion coefficients** of the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = {\Gamma^1}_{jk}\mathbf{e}_1 + {\Gamma^2}_{jk}\mathbf{e}_2 + {\Gamma^3}_{jk}\mathbf{e}_3 = {\Gamma^i}_{jk}\mathbf{e}_i,$$

  we can also expand the vector $\frac{\partial \mathbf{e}_j}{\partial x^k}$ with respect to the **dual basis $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$**.

- The new coefficients are denoted by Γ_{ijk}. That is

$$\frac{\partial \mathbf{e}_j}{\partial x^k} = \Gamma_{1jk}\mathbf{e}^1 + \Gamma_{2jk}\mathbf{e}^2 + \Gamma_{3jk}\mathbf{e}^3 = \Gamma_{ijk}\mathbf{e}^i.$$

- These new symbols are called the **Christoffel symbols of the first kind**.

PROPERTIES OF CHRISTOFFEL SYMBOLS

Properties of Christoffel symbols

- **Symmetry in the last two indices:** The Christoffel symbols ${\Gamma^i}_{jk}$ and Γ_{ijk} are symmetric in the indices j and k. That is

$$\Gamma_{ijk} = \Gamma_{ikj}, \qquad \text{and} \qquad {\Gamma^i}_{jk} = {\Gamma^i}_{kj}.$$

- They are **not tensors**!

- We have a formulae relating Christoffel symbols to the **metric tensor**:

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right),$$

$$\Gamma^i_{\cdot\, jk} = \frac{1}{2} g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right).$$

RICCI'S THEOREM:

Ricci's Theorem.

The covariant derivative of the metric tensor g_{ik} vanishes.

Useful formula

Ricci's theorem yields the following useful formula

$$\frac{\partial g_{ik}}{\partial x^{\ell}} = \Gamma_{ik\ell} + \Gamma_{ki\ell}.$$

In particular, in **orthogonal coordinates**

$$\Gamma_{ik\ell} + \Gamma_{ki\ell} = 0 \implies \Gamma_{ik\ell} = -\Gamma_{ki\ell}.$$

RIEMANN-CHRISTOFFEL TENSOR

DEFINING SPACES

Defining spaces

So far we have only been (mostly) dealing with the usual [1]Euclidean space $\mathbb{R}^3$, whether we used Cartesian coordinates or generalised coordinates.

However there are other types of spaces.

For example: the surface of a sphere is a two-dimensional non-Euclidean space.

Question: How can we distinguish between different spaces?

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[1] A Euclidean vector space is a finite-dimensional inner product space over the real numbers.

CURVATURE

Curvature

One way of distinguishing these spaces is to use **curvature**:

- Euclidean space has zero curvature,
- The surface of a sphere has positive curvature,
- Hyperbolic space has negative curvature.

Question: Is there a condition that tells us when a space must be Euclidean?

In other words, can we determine whether a space has zero curvature or not?

PARTIAL DERIVATIVES

Partial derivatives

Let **A** be a vector field with components $A_i(x^1, x^2, x^3)$.

- The partial derivative of a component is written as
$$\frac{\partial A_i}{\partial x^j}.$$
- Since each component $A_i = A_i(x^1, x^2, x^3)$ is a function, we are simply taking a **partial derivatice** of a function.
- For smooth functions, partial derivatives always commute
$$\frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\partial A_i}{\partial x^k}\right).$$

PARTIAL DERIVATIVES VS TENSORS

A key issue

- A vector field is not merely a collection of component functions.
- It is a geometric object whose components change under a change of coordinates. (I.e. a tensor.)
- If we compute

$$\frac{\partial A_i}{\partial x^j},$$

  for each component, the result does **not** transform as a tensor under general coordinate changes.
- Therefore, to differentiate vector fields in a way that respects the tensor properties, we need **a different notion of derivative**.

COVARIANT DERIVATIVES

Covariant derivatives

- To make distinction between both concepts, recall that we denote **the covariant derivative of** A_i by $A_{i,j}$.
- To stress out this distinction even more, write

$$A_{i,j} := \nabla_j A_i.$$

- We have seen that

$$A_{i,j} := \nabla_j A_i = \frac{\partial A_i}{\partial x^j} - \Gamma^k_{ij} A_k. \qquad \text{(Week 26.2)}$$

  - The partial derivative differentiates the components.
  - The Christoffel symbols correct for the change of basis.
  - The result transforms as a tensor.

THE EUCLIDEAN SPECIAL CASE

Euclidean space in Cartesian coordinates

In Euclidean space with Cartesian coordinates,

$$\nabla_j A_i = \frac{\partial A_i}{\partial x^j}.$$

This is because

$$\Gamma^k_{ij} = 0.$$

This is why partial and covariant derivatives appear identical in vector calculus.

SECOND DERIVATIVES

Second derivatives

Second partial derivatives always commute:

$$\frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\partial A_i}{\partial x^k}\right).$$

Second covariant derivatives are written as

$$A_{i,\,kj} := \nabla_k \nabla_j A_i.$$

There is no reason for these to commute in general. That is

$$\nabla_k \nabla_j A_i \neq \nabla_j \nabla_k A_i,$$

equivalently

$$A_{i,\,jk} \neq A_{i,\,kj}.$$

CURVATURE

Next steps

- We will show that $A_{i,jk} - A_{i,kj}$ is a tensor.
- This tensor is called **Riemann-Christoffel tensor**.
- To check if we are in **Euclidean space**, we check if the space has zero curvature.
- By the above, this amounts to verifying whether the **Riemann-Christoffel** tensor is zero.

Summary

- Partial derivatives always commute.
- Covariant derivatives commute **iff** curvature is zero.
- **Curvature** measures the failure of covariant derivatives to commute.

RIEMANN–CHRISTOFFEL TENSOR

Riemann–Christoffel

- To check whether we are in **Euclidean space**, we check whether the space has zero curvature.
- That is, whether covariant derivatives commute:

$$\nabla_j \nabla_k A_i = \nabla_k \nabla_j A_i.$$

- We define the **Riemann–Christoffel tensor** by

$$R^r_{ijk} A_r = \nabla_j \nabla_k A_i - \nabla_k \nabla_j A_i.$$

- We now derive a coordinate formula for R^r_{ijk}.

DERIVING THE RIEMANN–CHRISTOFFEL TENSOR

Deriving the Riemann–Christoffel tensor

Recall the general formula for covariant differentiation of a second-rank tensor

$$\nabla_j T_{ik} = \frac{\partial T_{ik}}{\partial x^j} - \Gamma^p_{ij} T_{pk} - \Gamma^p_{kj} T_{ip}.$$

We apply this to $T_{ik} = \nabla_k A_i$, obtaining

$$\nabla_j \nabla_k A_i = \frac{\partial(\nabla_k A_i)}{\partial x^j} - \Gamma^p_{ij} \nabla_k A_p - \Gamma^p_{kj} \nabla_p A_i.$$

DERIVING THE RIEMANN–CHRISTOFFEL TENSOR – PART 2

Deriving the Riemann–Christoffel tensor

Again

$$\nabla_j \nabla_k A_i = \frac{\partial(\nabla_k A_i)}{\partial x^j} - \Gamma^p_{ij} \nabla_k A_p - \Gamma^p_{kj} \nabla_p A_i.$$

Recall that

$$\nabla_k A_i = \frac{\partial A_i}{\partial x^k} - \Gamma^r_{ki} A_r.$$

Substituting,

$$\nabla_j \nabla_k A_i = \frac{\partial}{\partial x^j} \left(\frac{\partial A_i}{\partial x^k} - \Gamma^r_{ki} A_r \right) - \Gamma^p_{ij} \left(\frac{\partial A_p}{\partial x^k} - \Gamma^r_{kp} A_r \right) \\ - \Gamma^p_{kj} \left(\frac{\partial A_i}{\partial x^p} - \Gamma^r_{pi} A_r \right).$$

DERIVING THE RIEMANN–CHRISTOFFEL TENSOR – PART 3

Deriving the Riemann–Christoffel tensor

Expanding,

$$\begin{aligned}\nabla_j \nabla_k A_i &= \frac{\partial}{\partial x^j}\left(\frac{\partial A_i}{\partial x^k} - \Gamma^r_{ki} A_r\right) - \Gamma^p_{ij}\left(\frac{\partial A_p}{\partial x^k} - \Gamma^r_{kp} A_r\right) \\ &\quad - \Gamma^p_{kj}\left(\frac{\partial A_i}{\partial x^p} - \Gamma^r_{pi} A_r\right) \\ &= \frac{\partial^2 A_i}{\partial x^j \partial x^k} - \Gamma^r_{ki}\frac{\partial A_r}{\partial x^j} - A_r \frac{\partial \Gamma^r_{ki}}{\partial x^j} - \Gamma^p_{ij}\frac{\partial A_p}{\partial x^k} - \Gamma^p_{kj}\frac{\partial A_i}{\partial x^p} \\ &\quad + \left(\Gamma^p_{ij}\Gamma^r_{kp} + \Gamma^p_{kj}\Gamma^r_{pi}\right) A_r.\end{aligned}$$

Interchanging j and k gives the formula for $\nabla_k \nabla_j A_i$.

DERIVING THE RIEMANN–CHRISTOFFEL TENSOR – PART 4

Deriving the Riemann–Christoffel tensor

Recall that $\nabla_j \nabla_k A_i$ is given by

$$\frac{\partial^2 A_i}{\partial x^j \partial x^k} - \Gamma^r_{\ ki} \frac{\partial A_r}{\partial x^j} - A_r \frac{\partial \Gamma^r_{\ ki}}{\partial x^j} - \Gamma^p_{\ ij} \frac{\partial A_p}{\partial x^k} - \Gamma^p_{\ kj} \frac{\partial A_i}{\partial x^p} + \Gamma^p_{\ ij} \Gamma^r_{\ pk} A_r + \Gamma^p_{\ kj} \Gamma^r_{\ pi} A_r.$$

We find a formula for

$$\nabla_j \nabla_k A_i - \nabla_k \nabla_j A_i,$$

by observing each of the following different types of terms:

- terms involving second derivatives of A,
- terms involving first derivatives of A,
- terms depending only on the Christoffel symbols.

DERIVING THE RIEMANN–CHRISTOFFEL TENSOR – PART 5

Second derivatives

Recall that $\nabla_j \nabla_k A_i$ is given by

$$\boxed{\frac{\partial^2 A_i}{\partial x^j \partial x^k}} - \Gamma^r_{\ ki} \frac{\partial A_r}{\partial x^j} - A_r \frac{\partial \Gamma^r_{\ ki}}{\partial x^j} - \Gamma^p_{\ ij} \frac{\partial A_p}{\partial x^k} - \Gamma^p_{\ kj} \frac{\partial A_i}{\partial x^p} + \Gamma^p_{\ ij} \Gamma^r_{\ pk} A_r + \Gamma^p_{\ kj} \Gamma^r_{\ pi} A_r.$$

Thus, the terms of $\nabla_j \nabla_k A_i - \nabla_k \nabla_j A_i$ involving **second derivatives of** A are

$$\frac{\partial^2 A_i}{\partial x^j \partial x^k} - \frac{\partial^2 A_i}{\partial x^k \partial x^j}.$$

Since partial derivatives commute, this term vanishes:

$$\frac{\partial^2 A_i}{\partial x^j \partial x^k} - \frac{\partial^2 A_i}{\partial x^k \partial x^j} = 0.$$

Deriving the Riemann–Christoffel tensor – Part 6

First derivatives

Recall that $\nabla_j \nabla_k A_i$ is given by

$$\frac{\partial^2 A_i}{\partial x^j \partial x^k} \boxed{-\Gamma^r{}_{ki}\frac{\partial A_r}{\partial x^j}} - A_r \frac{\partial \Gamma^r{}_{ki}}{\partial x^j} \boxed{-\Gamma^p{}_{ij}\frac{\partial A_p}{\partial x^k} - \Gamma^p{}_{kj}\frac{\partial A_i}{\partial x^p}} + \Gamma^p{}_{ij}\Gamma^r{}_{pk}A_r + \Gamma^p{}_{kj}\Gamma^r{}_{pi}A_r.$$

Thus, the terms of $\nabla_j \nabla_k A_i - \nabla_k \nabla_j A_i$ involving **first derivatives of A** are

$$\Big(-\Gamma^r{}_{ki}\frac{\partial A_r}{\partial x^j} - \Gamma^p{}_{ij}\frac{\partial A_p}{\partial x^k} - \Gamma^p{}_{kj}\frac{\partial A_i}{\partial x^p}\Big) - \Big(-\Gamma^r{}_{ji}\frac{\partial A_r}{\partial x^k} - \Gamma^p{}_{ik}\frac{\partial A_p}{\partial x^j} - \Gamma^p{}_{jk}\frac{\partial A_i}{\partial x^p}\Big).$$

Rearranging,

$$\underbrace{\left(-\Gamma^r{}_{ki}\frac{\partial A_r}{\partial x^j} + \Gamma^p{}_{ik}\frac{\partial A_p}{\partial x^j}\right)}_{=0,\ \text{because } \Gamma^\ell{}_{ik}=\Gamma^\ell{}_{ki}} + \underbrace{\left(-\Gamma^p{}_{ij}\frac{\partial A_p}{\partial x^k} + \Gamma^r{}_{ji}\frac{\partial A_r}{\partial x^k}\right)}_{=0,\ \text{because } \Gamma^\ell{}_{ij}=\Gamma^\ell{}_{ji}} + \underbrace{\left(-\Gamma^p{}_{kj}\frac{\partial A_i}{\partial x^p} + \Gamma^p{}_{jk}\frac{\partial A_i}{\partial x^p}\right)}_{=0,\ \text{because } \Gamma^\ell{}_{kj}=\Gamma^\ell{}_{jk}}.$$

DERIVING THE RIEMANN–CHRISTOFFEL TENSOR – PART 7

Remaining terms

Recall that $\nabla_j \nabla_k A_i$ is given by

$$\frac{\partial^2 A_i}{\partial x^j \partial x^k} - \Gamma^r{}_{ki}\frac{\partial A_r}{\partial x^j} \boxed{-A_r \frac{\partial \Gamma^r{}_{ki}}{\partial x^j}} - \Gamma^p{}_{ij}\frac{\partial A_p}{\partial x^k} - \Gamma^p{}_{kj}\frac{\partial A_i}{\partial x^p} \boxed{+\Gamma^p{}_{ij}\Gamma^r{}_{pk}A_r + \Gamma^p{}_{kj}\Gamma^r{}_{pi}A_r}.$$

Since all terms involving first and second derivatives of A cancel out, we are left with terms depending only on the **Christoffel symbols**:

$$\nabla_j \nabla_k A_i - \nabla_k \nabla_j A_i = \left(\frac{\partial \Gamma^r{}_{ki}}{\partial x^j} - \frac{\partial \Gamma^r{}_{ij}}{\partial x^k}\right) A_r + \left(\Gamma^p{}_{ik}\Gamma^r{}_{pj} - \Gamma^p{}_{ij}\Gamma^r{}_{pk}\right) A_r.$$

Here $\frac{\partial \Gamma^r{}_{ki}}{\partial x^j}$ denotes the ordinary partial derivative of the function $\Gamma^r{}_{ki} = \Gamma^r{}_{ki}(x^1, x^2, x^3)$.

DERIVING THE RIEMANN–CHRISTOFFEL TENSOR – FINAL

Definition of curvature

We therefore define the **Riemann–Christoffel tensor** by

$$\nabla_j \nabla_k A_i - \nabla_k \nabla_j A_i = R^r_{\ ijk} A_r,$$

where

$$R^r_{\ ijk} = \frac{\partial \Gamma^r_{\ ki}}{\partial x^j} - \frac{\partial \Gamma^r_{\ ij}}{\partial x^k} + \Gamma^p_{\ ik} \Gamma^r_{\ pj} - \Gamma^p_{\ ij} \Gamma^r_{\ pk}.$$

THE RIEMANN–CHRISTOFFEL TENSOR IS A TENSOR

The Riemann–Christoffel tensor is in fact a tensor

Recall
$$\nabla_j \nabla_k A_i - \nabla_k \nabla_j A_i = R^r_{ijk} A_r. \tag{1}$$

Let us show that R^r_{ijk} is a tensor.

Proof.

- Since the left-hand side of (1) is a subtraction of tensors, it is a tensor.
- Since the left-hand side of (1) is a tensor, so is $R^r_{ijk} A_r$.
- Because A_r is arbitrary, it follows by the **quotient rule** that

$$R^r_{ijk} \quad \textbf{is a tensor}.$$

□

CONDITION FOR EUCLIDEAN SPACE

Condition for Euclidean Space

For **Euclidean space**, we need zero curvature: $A_{i,jk} = A_{i,kj}$.

We have shown

$$A_{i,jk} - A_{i,kj} = R^{r}_{ijk} A_r,$$

so the condition for a space to be **Euclidean** is therefore

$$A_{i,jk} = A_{i,kj} \iff A_{i,jk} - A_{i,kj} = 0 \iff R^{r}_{ijk} = 0.$$

This means that a Euclidean space must have a vanishing **Riemann-Christoffel** tensor.

EXAMPLE

Example

Consider the surface of the sphere of **radius $\mathbf{r} = \mathbf{1}$**, which is a surface of positive curvature.

The position vector is given in spherical coordinates $(x^1, x^2) = (\phi, \theta)$ by

$$\mathbf{r} = \sin\phi\cos\theta\,\mathbf{i}_1 + \sin\phi\sin\theta\,\mathbf{i}_2 + \cos\phi\,\mathbf{i}_3,$$

and thus

$$\begin{aligned}
\mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial \phi} = \cos\phi\cos\theta\,\mathbf{i}_1 + \cos\phi\sin\theta\,\mathbf{i}_2 - \sin\phi\,\mathbf{i}_3 \\
\mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial \theta} = -\sin\phi\sin\theta\mathbf{i}_1 + \sin\phi\cos\theta\,\mathbf{i}_2.
\end{aligned}$$

We will now find the components of the Riemann-Christoffel tensor.

EXAMPLE - PART 2

Example

Methodology: Recall Riemann-Christoffel tensor

$$R^r{}_{ijk} = \left[\frac{\partial}{\partial x^j} \Gamma^r{}_{ki} - \frac{\partial}{\partial x^k} \Gamma^r{}_{ij} + \Gamma^p{}_{ik} \Gamma^r{}_{pj} - \Gamma^p{}_{ij} \Gamma^r{}_{pk} \right].$$

So we will need:

- Christoffel symbols of second and first kind

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{i\ell} \left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell} \right),$$

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right).$$

- the metric tensor

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j.$$

EXAMPLE - PART 3

Example

We first find the components of the metric tensor via

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j.$$

Using the values we found for $\mathbf{e}_1$ and $\mathbf{e}_2$

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial \phi} = \cos\phi \cos\theta \, \mathbf{i}_1 + \cos\phi \sin\theta \, \mathbf{i}_2 - \sin\phi \, \mathbf{i}_3$$

$$\mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial \theta} = -\sin\phi \sin\theta \mathbf{i}_1 + \sin\phi \cos\theta \, \mathbf{i}_2.$$

we get

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = \sin^2\phi.$$

EXAMPLE - PART 4

Example

Next we find Christoffel symbols of the first kind using the formula

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right).$$

Again,

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = \sin^2\phi.$$

Thus, the only $\frac{\partial g_{ab}}{\partial x^c}$ that is non-zero is

$$\frac{\partial g_{22}}{\partial x^1} = \frac{\partial \sin^2\phi}{\partial \phi} = 2\sin\phi\cos\phi.$$

We conclude that $\Gamma_{ijk} = 0$, unless two indices are 2 and one index is 1.

EXAMPLE - PART 5

Example

Consequently,

$$\Gamma_{111} = \Gamma_{112} = \Gamma_{121} = \Gamma_{112} = \Gamma_{222} = 0.$$

Moreover,

$$\begin{aligned}\Gamma_{122} &= \frac{1}{2}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right)\\ &= \frac{1}{2}\left(-\frac{\partial \sin^2\phi}{\partial \phi}\right) = -\sin\phi\cos\phi.\end{aligned}$$

Using Ricci's Theorem, we obtain

$$0 = \frac{\partial g_{12}}{\partial x^2} = \Gamma_{122} + \Gamma_{212}.$$

We conclude

$$\Gamma_{212} = -\Gamma_{122} = \sin\phi\cos\phi.$$

EXAMPLE - PART 6

Example

Also, by the symmetry of the Chritoffel symbols $\Gamma_{221} = \Gamma_{212}$.

We conclude

$$\begin{aligned}
&\Gamma_{111} = \Gamma_{112} = \Gamma_{121} = \Gamma_{211} = \Gamma_{222} = 0,\\
&\Gamma_{122} = -\sin\phi\cos\phi,\\
&\Gamma_{221} = \Gamma_{212} = \sin\phi\cos\phi.
\end{aligned}$$

Next, we find Christoffel symbols of the second kind using

$$\Gamma^{i}{}_{jk} = g^{i\ell}\Gamma_{\ell jk}.$$

EXAMPLE - PART 7

Example

First, we find the components of the metric tensor

$$g^{11} = \frac{1}{g_{11}} = 1, \qquad g^{22} = \frac{1}{g_{22}} = \frac{1}{\sin^2\phi}, \qquad g^{12} = 0.$$

Recall that Christoffel symbols of the first kind are

$$\begin{aligned}&\Gamma_{111} = \Gamma_{112} = \Gamma_{121} = \Gamma_{211} = \Gamma_{222} = 0,\\ &\Gamma_{122} = -\sin\phi\cos\phi, \qquad \Gamma_{221} = \sin\phi\cos\phi.\end{aligned}$$

Thus, the Christoffel symbols of the second kind are

$$\begin{aligned}&\Gamma^1{}_{11} = \Gamma^1{}_{12} = \Gamma^2{}_{11} = \Gamma^2{}_{22} = 0,\\ &\Gamma^1{}_{22} = g^{1\ell}\Gamma_{\ell 22} = g^{11}\Gamma_{122} + g^{12}\Gamma_{222} = -\sin\phi\cos\phi,\\ &\Gamma^2{}_{12} = g^{2\ell}\Gamma_{\ell 12} = g^{21}\Gamma_{112} + g^{22}\Gamma_{212} = \cot\phi.\end{aligned}$$

EXAMPLE - PART 8

Example

The Christoffel symbols of the second kind are

$$\Gamma^1{}_{11} = \Gamma^1{}_{12} = \Gamma^1{}_{21} = \Gamma^2{}_{11} = \Gamma^2{}_{22} = 0,$$
$$\Gamma^1{}_{22} = -\sin\phi\cos\phi, \quad \Gamma^2{}_{12} = \Gamma^2_{21} = \cot\phi.$$

Therefore, we can find the Riemann-Christoffel tensor using the formula

$$R^r_{ijk} = \left[\frac{\partial}{\partial x^j}\Gamma^r{}_{ki} - \frac{\partial}{\partial x^k}\Gamma^r{}_{ij} + \Gamma^p{}_{ik}\Gamma^r{}_{pj} - \Gamma^p{}_{ij}\Gamma^r{}_{pk}\right].$$

Notice that

$$R^1{}_{111} = R^1{}_{112} = R^1{}_{121} = R^1{}_{211} = R^1{}_{122} = R^1{}_{221} = R^1{}_{222} = 0,$$
$$R^2{}_{111} = R^2{}_{211} = R^2{}_{221} = R^2{}_{122} = R^2{}_{222} = 0.$$

EXAMPLE - PART 9

Example

The Christoffel symbols of the second kind are

$$\begin{aligned} &{\Gamma^1}_{11} = {\Gamma^1}_{12} = {\Gamma^2}_{11} = {\Gamma^2}_{22} = 0, \\ &{\Gamma^1}_{22} = -\sin\phi\cos\phi, \quad {\Gamma^2}_{12} = \cot\phi. \end{aligned}$$

Moreover,

$$\begin{aligned} {R^1}_{212} &= \left[\frac{\partial}{\partial x^1}{\Gamma^1}_{22} - \frac{\partial}{\partial x^2}{\Gamma^1}_{21} + {\Gamma^p}_{22}{\Gamma^1}_{p1} - {\Gamma^p}_{21}{\Gamma^1}_{p2}\right] \\ &= \frac{\partial}{\partial\phi}{\Gamma^1}_{22} - {\Gamma^2}_{21}{\Gamma^1}_{22} \\ &= \frac{\partial}{\partial\phi}(-\sin\phi\cos\phi) - \frac{\cos\phi}{\sin\phi}(-\sin\phi\cos\phi) \\ &= -\cos^2\phi + \sin^2\phi + \cos^2\phi \\ &= \sin^2\phi. \end{aligned}$$

NEXT LECTURE

Next time...

Chapter 7:

- Ricci tensor.