

TENSOR ANALYSIS

SLIDES WEEK 31 – LECTURE 1

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CHAPTER 7: TENSOR FIELDS

Today: Chapter 7–Tensor Fields

1. Preliminary,
2. Covariant differentiation,
3. Christoffel symbols,
4. Covariant differentiation of tensors,
5. Ricci's theorem,
6. Riemann-Christoffel tensor,
7. Ricci tensor.

REMINDER

Generalised Coordinates with local basis

- Recall: We are now considering generalised coordinate systems with **local bases**.
- A **local basis** is a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ that varies from point to point:

$$\mathbf{e}_j = \mathbf{e}_j(x^1, x^2, x^3), \quad \mathbf{e}^j = \mathbf{e}^j(x^1, x^2, x^3).$$

Christoffel symbols

- We have a formulae relating Christoffel symbols to the **metric tensor**:

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right),$$

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right).$$

Recall

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j, \quad g^{ij} = \mathbf{e}^i \cdot \mathbf{e}^j.$$

- **Symmetry in the last two indices:** The Christoffel symbols $\Gamma^i{}_{jk}$ and Γ_{ijk} are symmetric in the indices j and k . That is

$$\Gamma_{ijk} = \Gamma_{ikj}, \quad \text{and} \quad \Gamma^i{}_{jk} = \Gamma^i{}_{kj}.$$

RICCI'S THEOREM:

Ricci's Theorem.

The covariant derivative of the metric tensor g_{ik} vanishes.

Useful formula

Ricci's theorem yields the following useful formula

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{ikl} + \Gamma_{kil}.$$

In particular, in **orthogonal coordinates**

$$\Gamma_{ikl} + \Gamma_{kil} = 0 \implies \Gamma_{ikl} = -\Gamma_{kil}.$$

Riemann–Christoffel

- To check whether we are in **Euclidean space**, we check whether the space has zero curvature.
- That is, whether covariant derivatives commute:

$$\nabla_j \nabla_k A_i = \nabla_k \nabla_j A_i.$$

- We define the **Riemann–Christoffel tensor** by

$$R^r_{ijk} A_r = \nabla_j \nabla_k A_i - \nabla_k \nabla_j A_i.$$

- We have shown

$$R^r_{ijk} = \frac{\partial \Gamma^r_{ki}}{\partial x^j} - \frac{\partial \Gamma^r_{ij}}{\partial x^k} + \Gamma^p_{ik} \Gamma^r_{pj} - \Gamma^p_{ij} \Gamma^r_{pk}.$$

Riemann-Christoffel tensor properties

- R^r_{ijk} is a tensor.
- A space is Euclidean if and only if the **Riemann-Christoffel** tensor vanishes.

RICCI TENSOR

Definition.

The **Ricci tensor** is defined as a trace of the Riemann-Christoffel tensor, that is

$$R_{ij} = R^k{}_{ikj}.$$

Riemann-Christoffel tensor and the Christoffel symbols

In terms of the Riemann-Christoffel tensor and the Christoffel symbols, one has

$$\begin{aligned} R_{ij} &= R^k{}_{ikj} \\ &= \frac{\partial}{\partial x^k} \Gamma^k{}_{ji} - \frac{\partial}{\partial x^j} \Gamma^k{}_{ik} + \Gamma^p{}_{ij} \Gamma^k{}_{pk} - \Gamma^p{}_{ik} \Gamma^k{}_{pj} \end{aligned}$$

This is a **symmetric** tensor.

In the next slides, we will solve the following.

Question

In this question, consider the three-dimensional coordinate system $(x^1, x^2, x^3) = (\rho, \theta, z)$ with position \mathbf{r} given by

$$\mathbf{r} = \rho \cos \theta \mathbf{i}_1 + \rho \sin \theta \mathbf{i}_2 + z \mathbf{i}_3,$$

where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are the usual Cartesian basis vectors.

Find the **Ricci tensor** with respect to this coordinate system.

Revision

- This exercise will revise many concepts of the previous weeks.
- The Ricci tensor is a trace of the Riemann-Christoffel tensor, that is

$$R_{ij} = R^k{}_{ikj}.$$

- Thus, we need to compute the components of the **Riemann-Christoffel tensor** $R^r{}_{ijk}$.
- In turn, to find the components of the Riemann-Christoffel tensor, we need the **Christoffel symbols** $\Gamma^i{}_{jk}$.
- The Christoffel symbols depend on the **metric coefficients** g_{ij} and g^{ij} .
- To find the metric coefficients, we need the **basis vectors** $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

Basis vectors

To find the basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of the given coordinate system, we use the formula

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i}.$$

Again, we are taking the coordinate system $(x^1, x^2, x^3) = (\rho, \theta, z)$ with position \mathbf{r} given by

$$\mathbf{r} = \rho \cos \theta \mathbf{i}_1 + \rho \sin \theta \mathbf{i}_2 + z \mathbf{i}_3.$$

Thus

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial x^1} = \frac{\partial \mathbf{r}}{\partial \rho} = \cos \theta \mathbf{i}_1 + \sin \theta \mathbf{i}_2,$$

$$\mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial x^2} = \frac{\partial \mathbf{r}}{\partial \theta} = -\rho \sin \theta \mathbf{i}_1 + \rho \cos \theta \mathbf{i}_2,$$

$$\mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial x^3} = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{i}_3.$$

Metric coefficients

Let us compute the metric tensor components g_{ij} . Let us start with the ones where $i \neq j$.

$$\begin{aligned}g_{12} = \mathbf{e}_1 \cdot \mathbf{e}_2 &= \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\rho \sin \theta \\ \rho \cos \theta \\ 0 \end{pmatrix} = -\rho \cos \theta \sin \theta + \phi \cos \theta \sin \theta \\ &= 0,\end{aligned}$$

$$g_{13} = \mathbf{e}_1 \cdot \mathbf{e}_3 = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0,$$

$$g_{23} = \mathbf{e}_2 \cdot \mathbf{e}_3 = \begin{pmatrix} -\rho \sin \theta \\ \rho \cos \theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0.$$

Since $g_{ij} = g_{ji}$, this shows that $g_{ij} = 0$ whenever $i \neq j$.

Orthogonality

- A coordinate system with basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is **orthogonal** whenever $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ if $i \neq j$.
- We have shown that $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ whenever $i \neq j$, so we conclude that this coordinate system is **orthogonal**.
- Note that $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ whenever $i \neq j$ is equivalent to $g_{ij} = 0$ for $i \neq j$.

Metric coefficients - part 2

Let us now compute the metric tensor components of the form g_{ii} .

$$g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} = \cos^2 \theta + \sin^2 \theta = 1,$$

$$\begin{aligned} g_{22} &= \mathbf{e}_2 \cdot \mathbf{e}_2 = \begin{pmatrix} -\rho \sin \theta \\ \rho \cos \theta \end{pmatrix} \cdot \begin{pmatrix} -\rho \sin \theta \\ \rho \cos \theta \end{pmatrix} = \rho^2 \sin^2 \theta + \rho^2 \cos^2 \theta \\ &= \rho^2 (\sin^2 \theta + \cos^2 \theta) = \rho^2, \end{aligned}$$

$$g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 1.$$

Orthonormality

A coordinate system with basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is **orthonormal** whenever $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Equivalently whenever

$$g_{ij} = \delta_{ij}.$$

We showed that

- $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = 0$ whenever $i \neq j$.
- $g_{11} = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1$ and $g_{33} = \mathbf{e}_3 \cdot \mathbf{e}_3 = 1$.
- However, $g_{22} = \mathbf{e}_2 \cdot \mathbf{e}_2 = \rho^2 \neq 1$.

Thus, this system is **not** orthonormal.

ARC LENGTH (EXTRA!)

Arc length

Let us also describe the arc length element in terms of the metric coefficients.¹ Since the system is orthogonal and

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the arc length element is given by

$$\begin{aligned}(ds)^2 &= g_{11}dx^1dx^1 + g_{22}dx^2dx^2 + g_{33}dx^3dx^3 \\ &= (d\rho)^2 + (\rho d\theta)^2 + (dz)^2.\end{aligned}$$

Thus, the metric coefficients are

$$h_1 = 1, \quad h_2 = \rho, \quad h_3 = 1.$$

¹Note this is not needed to reach our goal!

Christoffel symbols of first kind

Let us find all Christoffel symbols Γ_{ijk} of the first kind for the given coordinate system. Recall

$$\Gamma_{ijk} = \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i} \right).$$

Since

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we see that the only non-zero $\frac{\partial g_{mn}}{\partial x^p}$ is

$$\frac{\partial g_{22}}{\partial x^1} = \frac{\partial \rho^2}{\partial \rho} = 2\rho.$$

Thus then only non-zero Γ_{ijk} are the ones with two indices 2 and one index 1. That is, Γ_{122} , Γ_{212} , and Γ_{221} .

Christoffel symbols of first kind

Again, the only non-zero $\frac{\partial g_{mn}}{\partial x^p}$ is

$$\frac{\partial g_{22}}{\partial x^1} = \frac{\partial \rho^2}{\partial \rho} = 2\rho.$$

That is, $\Gamma_{122}, \Gamma_{212} = \Gamma_{221}$ are the only non-zero Γ_{ijk} . Now

$$\Gamma_{122} = \frac{1}{2} \left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1} \right) = -\frac{1}{2} \frac{\partial g_{22}}{\partial \rho} = -\frac{1}{2}(2\rho) = -\rho.$$

By Ricci's Theorem,

$$\Gamma_{212} = -\Gamma_{122} = \rho.$$

By the symmetry of Christoffel symbols on the last two indices, we also have

$$\Gamma_{221} = \Gamma_{212} = \rho.$$

Christoffel symbols of first kind - Conclusion

We conclude

$$\Gamma_{122} = -\rho,$$

$$\Gamma_{212} = \rho,$$

$$\Gamma_{221} = \rho,$$

$$\Gamma_{ijk} = 0, \quad \text{for all other values of } i, j, k.$$

Christoffel symbols of second kind

Let us find the Christoffel symbols of second kind. We have

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{so} \quad [g^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore

$$\Gamma_{22}^1 = g^{1\ell} \Gamma_{\ell 22} = g^{11} \Gamma_{122} = 1 \cdot (-\rho) = -\rho,$$

$$\Gamma_{21}^2 = \Gamma_{12}^2 = g_{21}^2 \Gamma_{221} = \frac{1}{\rho^2} \rho = \frac{1}{\rho},$$

$$\Gamma_{jk}^i = 0, \text{ for all other values of } i, j, k.$$

Riemann-Christoffel tensor

To illustrate how to compute the components of the Riemann-Christoffel tensor, let us find R^2_{121} .

We found

$$\Gamma^1_{22} = -\rho, \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{\rho}, \quad \Gamma^i_{jk} = 0, \text{ for other } i, j, k.$$

Thus,

$$\begin{aligned} R^2_{121} &= \frac{\partial}{\partial x^2} \Gamma^2_{11} - \frac{\partial}{\partial x^1} \Gamma^2_{12} + \Gamma^p_{11} \Gamma^2_{p2} - \Gamma^p_{12} \Gamma^2_{p1} \\ &= -\frac{\partial}{\partial x^1} \Gamma^2_{12} + \Gamma^p_{11} \Gamma^2_{p2} - \Gamma^p_{12} \Gamma^2_{p1} \quad (\Gamma^2_{11} = 0). \end{aligned}$$

Riemann-Christoffel tensor

Again

$$\Gamma^1_{22} = -\rho, \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{\rho}, \quad \Gamma^i_{jk} = 0, \text{ for other } i, j, k.$$

Continuing

$$\begin{aligned} R^2_{121} &= -\frac{\partial}{\partial x^1} \Gamma^2_{12} + \Gamma^p_{11} \Gamma^2_{p2} - \Gamma^p_{12} \Gamma^2_{p1} \\ &= -\frac{\partial}{\partial x^1} \Gamma^2_{12} + \left(\cancel{\Gamma^1_{11} \Gamma^2_{12}} + \cancel{\Gamma^2_{11} \Gamma^2_{22}} + \cancel{\Gamma^3_{11} \Gamma^2_{32}} \right) - \Gamma^p_{12} \Gamma^2_{p1} \\ &= -\frac{\partial}{\partial x^1} \Gamma^2_{12} - \cancel{\Gamma^1_{12} \Gamma^2_{11}} - \Gamma^2_{12} \Gamma^2_{21} - \cancel{\Gamma^3_{12} \Gamma^2_{31}} \\ &= -\frac{\partial}{\partial x^1} \Gamma^2_{12} - \Gamma^2_{12} \Gamma^2_{21} = -\frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \right) - \frac{1}{\rho} \cdot \frac{1}{\rho} \\ &= \frac{1}{\rho^2} - \frac{1}{\rho^2} = 0. \end{aligned}$$

Ricci tensor

Let us find the Ricci tensor

$$R_{ij} = R^k_{ikj} = R^1_{i1j} + R^2_{i2j}.$$

Let us find all values of R^1_{i1j} and of R^2_{i2j} separately.

RICCI TENSOR - R^1_{i1j}

$$\Gamma^1_{22} = -\rho, \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{\rho}, \quad \Gamma^i_{jk} = 0, \text{ for other } i, j, k.$$

Ricci tensor R^1_{i1j}

First, we have

$$\begin{aligned} R^1_{i1j} &= \frac{\partial}{\partial x^j} \cancel{\Gamma^1_{1i}} - \frac{\partial}{\partial x^1} \Gamma^1_{ij} + \Gamma^p_{i1} \Gamma^1_{pj} - \Gamma^p_{ij} \cancel{\Gamma^1_{p1}} \\ &= -\frac{\partial}{\partial x^1} \Gamma^1_{ij} + \Gamma^p_{i1} \Gamma^1_{pj} \\ &= -\frac{\partial}{\partial x^1} \Gamma^1_{ij} + \cancel{\Gamma^1_{i1}} \cancel{\Gamma^1_{1j}} + \Gamma^2_{i1} \Gamma^1_{2j} \\ &= -\frac{\partial}{\partial x^1} \Gamma^1_{ij} + \Gamma^2_{i1} \Gamma^1_{2j} \\ &= \begin{cases} 0, & \text{if } i = 1 \text{ or } j = 1 \\ 1 + \frac{1}{\rho}(-\rho) = 0, & \text{otherwise.} \end{cases} \end{aligned}$$

RICCI TENSOR - R^2_{i2j} - CASE 1

$$\Gamma^1_{22} = -\rho, \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{\rho}, \quad \Gamma^i_{jk} = 0, \text{ for other } i, j, k.$$

Ricci tensor R^2_{i2j} - case 1

Moreover

$$\begin{aligned} R^2_{i2j} &= \frac{\partial}{\partial x^j} \Gamma^2_{2i} - \underbrace{\frac{\partial}{\partial x^2} \Gamma^2_{ij}}_{\frac{\partial}{\partial x^2} (\Gamma^i_{jk})=0} + \Gamma^p_{i2} \Gamma^2_{pj} - \Gamma^p_{ij} \Gamma^2_{p2} \\ &= \frac{\partial}{\partial x^j} \Gamma^2_{2i} + \Gamma^p_{i2} \Gamma^2_{pj} - \Gamma^p_{ij} \Gamma^2_{p2}. \end{aligned}$$

If $(i, j) = (1, 2)$ or $(i, j) = (2, 1)$, then

$$\begin{aligned} R^2_{i2j} &= \underbrace{\frac{\partial}{\partial x^j} \Gamma^2_{2i}}_{=0} + \Gamma^p_{i2} \Gamma^2_{pj} - \Gamma^p_{ij} \Gamma^2_{p2} \\ &= \underbrace{\Gamma^1_{i2} \Gamma^2_{1j}}_{=0} + \underbrace{\Gamma^2_{i2} \Gamma^2_{2j}}_{=0} - \cancel{\Gamma^1_{ij} \Gamma^2_{12}} - \cancel{\Gamma^2_{ij} \Gamma^2_{22}} = 0. \end{aligned}$$

$$\Gamma^1_{22} = -\rho, \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{\rho}, \quad \Gamma^i_{jk} = 0, \text{ for other } i, j, k.$$

Ricci tensor R^2_{i2j} - case 2

In the previous slide, we have shown

$$R^2_{122} = 0, \quad R^2_{221} = 0,$$

In a previous slide we already computed $R^2_{121} = 0$.

Thus to finish all R^2_{i2j} , we need to compute R^2_{222} .

$$\Gamma^1_{22} = -\rho, \quad \Gamma^2_{21} = \Gamma^2_{12} = \frac{1}{\rho}, \quad \Gamma^i_{jk} = 0, \text{ for other } i, j, k.$$

Ricci tensor R^2_{222}

Finally, since

$$R^2_{i2j} = \frac{\partial}{\partial x^j} \Gamma^2_{2i} + \Gamma^p_{i2} \Gamma^2_{pj} - \Gamma^p_{ij} \Gamma^2_{p2},$$

we get

$$\begin{aligned} R^2_{222} &= \frac{\partial}{\partial x^2} \cancel{\Gamma^2_{22}} + \Gamma^p_{22} \Gamma^2_{p2} - \Gamma^p_{22} \Gamma^2_{p2} \\ &\quad \Gamma^1_{22} \Gamma^2_{12} + \cancel{\Gamma^2_{22} \Gamma^2_{22}} - \Gamma^1_{22} \Gamma^2_{12} - \cancel{\Gamma^2_{22} \Gamma^2_{22}} = 0. \end{aligned}$$

Ricci tensor - Conclusion

We have found that for all $i, j \in \{1, 2\}$

$$R^1_{i1j} = 0,$$

$$R^2_{i2j} = 0.$$

Therefore,

$$R_{ij} = R^k_{ikj} = R^1_{i1j} + R^2_{i2j} = 0,$$

for all i, j .

Next time...

Revision

- Suffix and vector notation, delta and alternating tensor,
- Vector Differential Operators,
- Local coordinate transformation,
- Tensors in orthogonal (Cartesian) Coordinates