Tensor Analysis

Slides week 31 – Lecture 2

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Revision

Today: Revision

- Suffix and vector notation, delta and alternating tensor,
- Vector Differential Operators,
- Local coordinate transformation
- Tensors in orthogonal (Cartesian) Coordinates
- Dual basis
- Covariant and contravariant components of a vector
- Covariant and contravariant components of a 2^{nd}-rank tensor
- Tensors in a generalised coordinate system
- Symmetries
- Tensor algebra
- Arc length and the metric tensor
- Christoffel symbols & Ricci's Theorem
- Riemann-Christoffel tensor

Suffix and vector notation, delta and alternating tensor

Vector vs suffix notation

- Suffix notation is a **local** notation.
- Vector notation is a **global** notation.
- For instance
  - **Vector notation:** $\mathbf{v} = (v_1, v_2, v_3)$ (i.e. the whole vector).
  - **Suffix notation:** v_i (i.e. only one component)
- For a scalar λ
  - **Vector notation:** $\lambda\mathbf{v} = (\lambda v_1, \lambda v_2, \lambda v_3)$.
  - **Suffix notation:** λv_i.

Dummy indices and free indices

- **Free indices:** Index appearing exactly **once in each term** that indicates the **component of a tensor**.
  - E.g. v_i means the i-th component of the vector $\mathbf{v}$.
- **Dummy indices:** Indices appearing **exactly twice in a term** that indicate a **sum**.
  - E.g. the dot product

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{j=1}^{3} a_i b_i$$

can be written in suffix notation as

$$a_i b_i.$$

Kronecker delta

The **Kronecker delta** δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Properties.

- "Substitution tensor":

$$\begin{aligned} \delta_{ij} a_j &= a_i \quad \text{(i.e. repeated index gets swallowed up)} \\ \delta_{ji} a_i &= a_j \end{aligned}$$

- Dot product in terms of δ_{ij}:

$$\mathbf{a} \cdot \mathbf{b} = \delta_{ij} a_i b_j.$$

The Alternating Tensor

- The **Alternating tensor** is defined by

$$\epsilon_{ijk} = \begin{cases} 0 & \text{if any of } i, j, k \text{ are equal,} \\ +1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1) \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) = (1,3,2),\ (2,1,3) \text{ or } (3,2,1). \end{cases}$$

- It remains unchanged if indices are reordered by a **cyclic permutation**:

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij}.$$

- the sign changes if any two of the suffices are **interchanged**:

$$\epsilon_{ijk} = -\epsilon_{jik}.$$

Properties of the Alternating Tensor

The Alternating Tensor

- can be used to define the cross product:

$$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k.$$

- is related to the determinant of a matrix:

$$|M| = \epsilon_{ijk} M_{1i} M_{2j} M_{3k}, \qquad \text{(row)}$$
$$|M| = \epsilon_{ijk} M_{i1} M_{j2} M_{k3}. \qquad \text{(column)}$$

- can be used to define the scalar triple product:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \epsilon_{ijk} a_i b_j c_k.$$

Your turn!

Corsework 23-24

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be vectors. Use **suffix notation** to show the following equality:

$$((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}) \cdot \mathbf{d} = -(\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) + (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}).$$

Vector Differential Operators

Differential Operators

- **The gradient of a scalar field:**

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right).$$

Or, in suffix notation

$$[\nabla f]_i = \frac{\partial f}{\partial x_i}.$$

- **The divergence of a vector field:**

$$\nabla \cdot \mathbf{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_j}{\partial x_j} = \nabla_i u_i,$$

Differential Operators

- **The curl of a vector field:**

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix}$$

$$= \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right)$$

In suffix notation

$$(\mathbf{u} \times \mathbf{v})_i = \epsilon_{ijk} u_j v_k.$$

POSITION VECTOR AND DIFFERENTIAL OPERATORS

Position vector

The **position vector** is

$$\mathbf{r} = (x_1, x_2, x_3).$$

We denote its **magnitude** by

$$\underline{r} = |\mathbf{r}|.$$

Position vector and differential operators

We have computed the gradient, divergence and curl of $\mathbf{r}$.

- $\nabla \underline{r} = \frac{\mathbf{r}}{\underline{r}}$, where $\underline{r} = |\mathbf{r}|$,
- $\nabla \cdot \mathbf{r} = 3$, and
- $[\nabla \times \mathbf{r}]_i = (\mathbf{0})_i$, that is, $\nabla \times \mathbf{r} = (0, 0, 0)$.

Local coordinate transformation

Coordinate Systems

Coordinate systems are defined by a set of **basis vectors**.

Vector basis is a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ that

- are **linearly independent:** the only scalars $\lambda_1, \lambda_2, \lambda_3$ satisfying

  $$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 = (0, 0, 0)$$

  are $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

- **span the space**: Every vector $\mathbf{v}$ can be written as a linear combination

  $$\mathbf{v} = \mu_1 \mathbf{v}_1 + \mu_2 \mathbf{v}_2 + \mu_3 \mathbf{v}_3.$$

Orthogonal and Orthonormal coordinate systems

Definitions

- A coordinate system is **orthogonal** if its basis vectors intersect at 90° angles, i.e.,

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0, \text{ for } i \neq j.$$

- A coordinate system is **orthonormal** if it is orthogonal and its basis vectors have magnitude 1, i.e.,
  - $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, for $i \neq j$, and
  - $|\mathbf{v}_i| = \sqrt{\mathbf{v}_i \cdot \mathbf{v}_i} = 1.$

Usual Cartesian coordinate system

Example

The **usual Cartesian coordinate** system

$$\hat{\mathbf{i}} = (1, 0, 0), \quad \hat{\mathbf{j}} = (0, 1, 0), \quad \hat{\mathbf{k}} = (0, 0, 1)$$

is an example of **orthonormal coordinate system**.

Rotating a basis

- Let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be basis vectors.
- If we rotate these basis vectors, we will obtain new basis vectors $\mathbf{e}_1'$, $\mathbf{e}_2'$, and $\mathbf{e}_3'$.
- This yields a new coordinate system (x_1', x_2', x_3').
- How do we swtich from one coordinate system to another?

Switching between the coordinate systems

When we say "switch between the coordinate systems (x_1, x_2, x_3) and (x'_1, x'_2, x'_3)" we mean:

- Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be a basis for the coordinate system (x_1, x_2, x_3).
- Let $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ be a basis for the coordinate system (x'_1, x'_2, x'_3).
- By the definition of basis, we can expand **any** vector $\mathbf{v}$ in terms of these bases. That is,

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3, \text{ and}$$
$$\mathbf{v} = v'_1\mathbf{e}'_1 + v'_2\mathbf{e}'_2 + v'_3\mathbf{e}'_3,$$

  for some constants v_i and v'_i.

- How do we get v_k's from v'_k's and vice versa?

Transformation Matrix

- We have seen that if we define a matrix $L = [L_{ij}]$ with components
$$L_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j,$$
then we have
$$x'_i = L_{ij} x_j.$$

- This matrix is called the **transformation matrix** because it allows you to switch between the coordinate systems (x_1, x_2, x_3) and (x'_1, x'_2, x'_3).

Properties of the transformation matrix

- The number L_{ij} is simply the cosine of the angle between $\mathbf{e}'_i$ and $\mathbf{e}_j$.
- L is **orthogonal** i.e. $L^T L = I = LL^T$. In particular,

$$L_{ij}L_{kj} = \delta_{ik} = L_{ji}L_{jk}.$$

- Since $x'_i = L_{ij}x_j$ and $x_i = L_{ji}x'_j$, we have

$$\frac{\partial x'_i}{\partial x_j} = L_{ij} \qquad \text{and} \qquad \frac{\partial x_i}{\partial x'_j} = L_{ji}.$$

Transformation Matrix Example

Example

Let K be a Cartesian coordinate system with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$. Define K' with basis

$$\begin{aligned} \mathbf{e}_1 &= \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3, \\ \mathbf{e}_2 &= 2\mathbf{i}_2, \\ \mathbf{e}_3 &= \mathbf{i}_1 + 3\mathbf{i}_2. \end{aligned}$$

Let us find the rotation matrix $L = [L_{ij}]$.

Method 1: $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ is Cartesian, and

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{pmatrix}, \quad \text{thus } L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 3 & 0 \end{pmatrix}.$$

Transformation Matrix Example

Example

Method 2: Using definition.

Recall: $\mathbf{e}_1 = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3, \quad \mathbf{e}_2 = 2\mathbf{i}_2, \quad \mathbf{e}_3 = \mathbf{i}_1 + 3\mathbf{i}_2.$

$$
\begin{aligned}
L_{11} &= \mathbf{e}_1 \cdot \mathbf{i}_1 = 1, & L_{12} &= \mathbf{e}_1 \cdot \mathbf{i}_2 = 1, & L_{13} &= \mathbf{e}_1 \cdot \mathbf{i}_3 = 1, \\
L_{21} &= \mathbf{e}_2 \cdot \mathbf{i}_1 = 0, & L_{22} &= \mathbf{e}_2 \cdot \mathbf{i}_2 = 2, & L_{23} &= \mathbf{e}_2 \cdot \mathbf{i}_3 = 0, \\
L_{31} &= \mathbf{e}_3 \cdot \mathbf{i}_1 = 1, & L_{32} &= \mathbf{e}_3 \cdot \mathbf{i}_2 = 3, & L_{33} &= \mathbf{e}_3 \cdot \mathbf{i}_3 = 0.
\end{aligned}
$$

Tensors in othogonal (Cartesian) coordinate systems

Formal definition: transformation rule

A **vector** is a quantity satisfying

$$v'_i = L_{ij} v_j.$$

That is, a vector is a quantity which **transforms** in a certain way under a rotation of coordinates.

This is called the **transformation rule** (of a vector).

Transformation (rotation) matrix L_{ij}

The matrix L_{ij} has the special property $L^T = L^{-1}$. This allows us to show

$$L_{ij}(L_{jk})^T = L_{ik}L_{kj} = \delta_{ik}.$$

Formal definition

A quantity is a **tensor** if each of the free indices satisfy the tranformation rule under rotation of the coordinates.

For instance,

- **Rank 2:** $T'_{ij} = L_{im}L_{jn}T_{mn}$,
- **Rank 3:** $T'_{ijk} = L_{im}L_{jn}L_{kp}T_{mnp}$,
- **Rank 4:** $T'_{ijk\ell} = L_{im}L_{jn}L_{kp}L_{\ell q}T_{mnpq}$, etc.

Definition

The **rank** or **order** of a tensor is the number of free indices.

Your turn!

Portfolio Test 23-24

Suppose $T_{ijk\ell m}$ is a rank-five tensor. Using the formal definition (i.e. the transformation rule), show that T_{ijkjk} is a rank-one tensor (i.e. a vector).

Next time...

Revision

- Dual basis
- Covariant and contravariant components of a vector
- Covariant and contravariant components of a second-rank tensor
- Tensors in a generalised coordinate system
- Symmetries