# TENSOR ANALYSIS

## SLIDES WEEK 31 – LECTURE 2

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# REVISION

# Today: Revision

- Suffix and vector notation, delta and alternating tensor,
- Vector Differential Operators,
- Local coordinate transformation,
- Tensors in orthogonal (Cartesian) Coordinates,
- Dual basis,
- Covariant and contravariant components of a vector,
- Covariant and contravariant components of a $2^{\text{nd}}$-rank tensor,
- Tensors in a generalised coordinate system,
- Symmetries,
- Tensor algebra,
- Arc length and the metric tensor,
- Christoffel symbols & Ricci's Theorem,
- Riemann-Christoffel tensor,

# DUAL BASES

# GENERALISED COORDINATE SYSTEMS

## Definition

**Generalised coordinate systems** do not necessarily have orthogonal coordinate bases.

## Next

- How do we switch between two generalised coordinate systems?
- It is a bit more challenging than just rotating a orthogonal coordinate system.
- Let us start with a particular case: dual basis.

# DUAL BASES

## Definition.

Two bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}^1$, $\mathbf{e}^2$ , $\mathbf{e}^3$ are **dual** if they satisfy

$$\mathbf{e}_i \cdot \mathbf{e}^k = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

## Method of Dual Basis

The dual basis $\mathbf{e}^i$ can be found from the original basis $\mathbf{e}_i$ via

$$\mathbf{e}^i = \frac{\mathbf{e}_j \times \mathbf{e}_k}{\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k)} = \frac{\mathbf{e}_j \times \mathbf{e}_k}{V},$$

and the original basis $\mathbf{e}_i$ can be found from the dual basis $\mathbf{e}^i$ via

$$\mathbf{e}_i = \frac{\mathbf{e}^j \times \mathbf{e}^k}{\mathbf{e}^i \cdot (\mathbf{e}^j \times \mathbf{e}^k)} = \frac{\mathbf{e}^j \times \mathbf{e}^k}{V'},$$

where $(i, j, k)$ is a cyclic permutation of $(1, 2, 3)$.

# EXAMPLE - DUAL BASIS

## Example - Dual Basis

Let (again) $K$ be a Cartesian coordinate system with orthonormal basis $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$, and $K'$ is a coordinate system with basis

$$\mathbf{e}_1 = 2\mathbf{i}_1 + 2\mathbf{i}_2 + \mathbf{i}_3, \qquad \mathbf{e}_2 = \mathbf{i}_1 + \mathbf{i}_2, \qquad \mathbf{e}_3 = \mathbf{i}_1 + 2\mathbf{i}_2 + 4\mathbf{i}_3.$$

Let us find the dual basis $\mathbf{e}^i$.

First, we find the **cross products**:

$$\mathbf{e}_1 \times \mathbf{e}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

# EXAMPLE - DUAL BASIS - PART 2

## Example - Dual Basis

$K'$ has basis

$$\mathbf{e}_1 = 2\mathbf{i}_1 + 2\mathbf{i}_2 + \mathbf{i}_3, \qquad \mathbf{e}_2 = \mathbf{i}_1 + \mathbf{i}_2, \qquad \mathbf{e}_3 = \mathbf{i}_1 + 2\mathbf{i}_2 + 4\mathbf{i}_3.$$

Thus,

$$\mathbf{e}_2 \times \mathbf{e}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}$$

$$\mathbf{e}_3 \times \mathbf{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \\ -2 \end{pmatrix}$$

# EXAMPLE - DUAL BASIS - PART 3

## Example - Dual Basis

Now, we find $V$:

$$\begin{aligned} V &= \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) \\ &= \mathbf{e}_2 \cdot (\mathbf{e}_3 \times \mathbf{e}_1) \\ &= \mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2). \end{aligned}$$

So, for instance, we can compute

$$\mathbf{e}_3 \cdot (\mathbf{e}_1 \times \mathbf{e}_2) = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -1 + 2 + 0 = 1.$$

# Example - Dual Basis - Part 4

## Example - Dual Basis

We have computed $V = 1$ and

$$\mathbf{e}_1 \times \mathbf{e}_1 = (-1, 1, 0)^T\,, \quad \mathbf{e}_2 \times \mathbf{e}_3 = (4, -1, 1)^T\,, \quad \mathbf{e}_3 \times \mathbf{e}_1 = (-6, 7, -2)^T\,.$$

Thus, the dual basis vectors are

$$\mathbf{e}^1 = \frac{1}{V}(\mathbf{e}_2 \times \mathbf{e}_3) = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix},$$

$$\mathbf{e}^2 = \frac{1}{V}(\mathbf{e}_3 \times \mathbf{e}_1) = \begin{pmatrix} -6 \\ 7 \\ -2 \end{pmatrix},$$

$$\mathbf{e}^3 = \frac{1}{V}(\mathbf{e}_1 \times \mathbf{e}_2) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

# Covariant and contravariant components of a vector

# EXPANSIONS

## Expansions

A vector $\mathbf{A}$ can be expanded

1. with respect to the vectors of a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ via

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A^i\mathbf{e}_i$$

2. with respect to the vectors of the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ via

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3 = A_i\mathbf{e}^i.$$

# CONTRAVARIANT COMPONENT

## Contravariant component

A vector $\mathbf{A}$ can be expanded

1. with respect to the vectors of a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ via

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A^i\mathbf{e}_i$$

2. with respect to the vectors of the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ via

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3 = A_i\mathbf{e}^i$$

$A^i$ are called the **contravariant** components of $\mathbf{A}$

# COVARIANT COMPONENT

## Covariant component

A vector $\mathbf{A}$ can be expanded

1. with respect to the vectors of a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ via

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A^i\mathbf{e}_i$$

2. with respect to the vectors of the dual basis $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ via

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3 = A_i\mathbf{e}^i$$

$A_i$ are called the **covariant** components of $\mathbf{A}$

# EXAMPLE

## Example.

In the previous example, we were given a basis

$$\mathbf{e}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

and we computed the dual basis

$$\mathbf{e}^1 = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}, \qquad \mathbf{e}^2 = \begin{pmatrix} -6 \\ 7 \\ -2 \end{pmatrix}, \qquad \mathbf{e}^3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

Let us find the covariant and contravariant components of the vector $\mathbf{J}$ joining the origin to the point $(2, -1, 0)$.

# EXAMPLE - PART 2

## Example.

**Covariant** components of $\mathbf{J} = (2, -1, 0)^T$: $J_i = \mathbf{J} \cdot \mathbf{e}_i$.

We have basis

$$\mathbf{e}_1 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix},$$

Consequently

$$\begin{aligned} J_1 &= \mathbf{J} \cdot \mathbf{e}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 4 - 2 + 0 = 2 \\ J_2 &= \mathbf{J} \cdot \mathbf{e}_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2 - 1 + 0 = 1, \\ J_3 &= \mathbf{J} \cdot \mathbf{e}_3 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = 2 - 2 + 0 = 0. \end{aligned}$$

# EXAMPLE - PART 3

## Example.

**Contravariant** components of $\mathbf{J} = (2, -1, 0)^T$: $J^i = \mathbf{J} \cdot \mathbf{e}^i$.

We have dual basis

$$\mathbf{e}^1 = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}, \qquad \mathbf{e}^2 = \begin{pmatrix} -6 \\ 7 \\ -2 \end{pmatrix}, \qquad \mathbf{e}^3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix},$$

Thus, we have

$$J^1 = \mathbf{J} \cdot \mathbf{e}^1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix} = 8 + 4 + 0 = 12,$$

$$J^2 = \mathbf{J} \cdot \mathbf{e}^2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 7 \\ -2 \end{pmatrix} = -12 - 7 + 0 = -19,$$

$$J^3 = \mathbf{J} \cdot \mathbf{e}^3 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -2 - 1 + 0 = -3.$$

# Example - Part 4

## Example.

We can even check whether our solution is correct!

We found that $\mathbf{J} = (2, -1, 0)^T$ has

- **Covariant** components $J_1 = 2, \quad J_2 = 1, \quad J_3 = 0.$
- **Contravariant** components $J^1 = 9, \quad J^2 = -19, \quad J^3 = -3.$

This means that

$$\mathbf{J} = 2\mathbf{e}^1 + 1\mathbf{e}^2 + 0\mathbf{e}^3 = 12\mathbf{e}_1 - 19\mathbf{e}_2 - 3\mathbf{e}_3.$$

In fact,

$$2\mathbf{e}^1 + \mathbf{e}^2 = 2\begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix} + \begin{pmatrix} -6 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \mathbf{J}.$$

$$\begin{aligned} 12\mathbf{e}_1 - 19\mathbf{e}^2 - 3\mathbf{e}_3 &= 12\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 19\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 24 \\ 24 \\ 12 \end{pmatrix} + \begin{pmatrix} -19 \\ -19 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ -6 \\ -12 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \mathbf{J}. \end{aligned}$$

# Different coordinate systems

## Different coordinate systems

Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ be bases of two coordinate systems. A vector $\mathbf{A}$ can be written in terms of both coordinate systems:

$$\mathbf{A} = A^1\mathbf{e}_1 + A^2\mathbf{e}_2 + A^3\mathbf{e}_3 = A^i\mathbf{e}_i,$$

$$\mathbf{A} = A_1\mathbf{e}^1 + A_2\mathbf{e}^2 + A_3\mathbf{e}^3 = A_i\mathbf{e}^i.$$

$$\mathbf{A} = A'^1\mathbf{e}'{}_1 + A'^2\mathbf{e}'_2 + A'^3\mathbf{e}'_3 = A'^i\mathbf{e}'_i,$$

$$\mathbf{A} = A'_1\mathbf{e}'^1 + A'_2\mathbf{e}'^2 + A'_3\mathbf{e}'^3 = A'_i\mathbf{e}'^i.$$

We have nice formulae to find components in one coordinate system in terms of the other.

# THE TRANSFORMATION RULE

## Transformation matrix

The **transformation matrix** from the coordinate system with basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ to the one with basis $\mathbf{e}'_1$, $\mathbf{e}'_2$, , $\mathbf{e}'_3$ is the matrix $L = [L^k_{i'}]$ with components

$$L^k_{i'} = \mathbf{e}'_i \cdot \mathbf{e}^k.$$

## The transformation rule

The **transformation rule** for the **covariant** components is

$$A'_i = L^j_{i'} A_j.$$

The **transformation rule** for the **contravariant** components is

$$A'^i = L^{i'}_j A^j.$$

# COVARIANT TO CONTRAVARIANT COMPONENTS

## Covariant to contravariant components

To express the covariant components $A_i$ of a vector **A** in terms of is contravariant components $A^i$ (and vice versa) we use

$$A_i = g_{ik} A^k,$$
$$A^i = g^{ik} A_k,$$

where

$$g_{ik} = g_{ki} = \mathbf{e}_i \cdot \mathbf{e}_k,$$
$$g^{ik} = g^{ki} = \mathbf{e}^i \cdot \mathbf{e}^k.$$

# COVARIANT AND CONTRAVARIANT COMPONENTS OF A SECOND RANK TENSOR

# SECOND-RANK TENSORS - COVARIANT AND CONTRAVARIANT COMPONENTS

### Second-rank tensors - covariant and contravariant components

- **covariant** $A_{ik}$,
- **contravariant** $A^{ik}$,
- **mixed** $A_i^{.k}$, $A^i_{.k}$.

# SECOND-RANK TENSORS - TRANSFORMATIONS

## Second-rank tensors - transformations

The components are transformed via:

$$
\begin{aligned}
A'_{ik} &= L^{\ell}_{i'} L^{m}_{k'} A_{\ell m}, \\
A'^{ik} &= L^{i'}_{\ell} L^{k'}_{m} A^{\ell m}, \\
A'^{\cdot k}_{i} &= L^{\ell}_{i'} L^{k'}_{m} A^{\cdot m}_{\ell}, \\
A'^{i}_{\cdot k} &= L^{i'}_{\ell} L^{m}_{k'} A^{\ell}_{\cdot m},
\end{aligned}
$$

where $L^{k}_{i'}$ and $L^{k'}_{i}$ $(i, k = 1, 2, 3)$ are the coefficients of the direct and the inverse transformations.

# EXAMPLE

## Example

Consider again the coordinate systems $K$ and $K'$ with bases $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ and

$$\mathbf{e}_1 = 2\mathbf{i}_1 + 2\mathbf{i}_2 + \mathbf{i}_3, \qquad \mathbf{e}_2 = \mathbf{i}_1 + \mathbf{i}_2, \qquad \mathbf{e}_3 = \mathbf{i}_1 + 2\mathbf{i}_2 + 4\mathbf{i}_3.$$

Consider the second-order tensor of $K$ with components

$$[A_{ik}] = [A^{ik}] = [A_i^{\cdot k}] = [A^i_{\cdot k}] = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Express the covariant components of the given tensor in the coordinate system $K'$.

# EXAMPLE - PART 2

## Example

Let $A = [A_{ik}]$ and $L = [L^k_{i'}]$. Notice that

$$\begin{aligned} L^\ell_{i'} L^m_{k'} A_{\ell m} &= L^\ell_{i'} A_{\ell m} L^m_{k'} \\ &= L^\ell_{i'} A_{\ell m} L^m_{k'} \\ &= (L\,A)_{im} L^m_{k'} \\ &= (L\,A)_{im} (L^T)^{k'}_m \\ &= (L\,A\,L^T)_{ik}. \end{aligned}$$

Write $A' = [A^{ik}]$. Using the formula $A'_{ik} = L^\ell_{i'} L^m_{k'} A_{\ell m}$, we see that

$$A' = L\,A\,L^T.$$

# EXAMPLE - PART 3

## Example

We then need to compute the transoformation matrix $L$.

$L$ is the matrix satisfying

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = L \begin{pmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{pmatrix}.$$

Since

$$\mathbf{e}_1 = 2\mathbf{i}_1 + 2\mathbf{i}_2 + \mathbf{i}_3,$$

$$\mathbf{e}_2 = \mathbf{i}_1 + \mathbf{i}_2,$$

$$\mathbf{e}_3 = \mathbf{i}_1 + 2\mathbf{i}_2 + 4\mathbf{i}_3,$$

we see that

$$L = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}.$$

# EXAMPLE - PART 3

## Example

Thus

$$\begin{aligned} A' = [A^{ik}] &= L\,A\,L^T \\ &= \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}^T \\ &= \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ -2 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 1 & 11 \\ 1 & 0 & 4 \\ 5 & 2 & 14 \end{pmatrix}. \end{aligned}$$

# EXAMPLE - CONTRAVARIANT AND MIXED COMPONENTS

## Example

Compute the covariant and mixed components of

$$[A_{ik}] = [A^{ik}] = [A_i^{\cdot k}] = [A^i_{\cdot k}] = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

in the coordinate system $K'$.

To find $A'^{ik}$, $A'^{.k}_{i}$ and $A'^{i}_{.k}$, we use the formulae

$$A'^{ik} = g^{i\ell} g^{km} A'_{\ell m},$$
$$A'^{.k}_{i} = g^{k\ell} A'_{i\ell},$$
$$A'^{i}_{.k} = g^{i\ell} A'_{\ell k},$$

# EXAMPLE - CONTRAVARIANT AND MIXED COMPONENTS - PART 2

## Example

Using the formula

$$A'^{ik} = g^{i\ell} g^{km} A'_{\ell m},$$

we get

$$\begin{aligned} A'^{ik} g^{i\ell} g^{km} A'_{\ell m} &= g^{i\ell} A'_{\ell m} g^{km} \\ &= (G\,A')_{im} g^{km} \\ &= (G\,A')_{im} G^T_{mk} \\ &= (G\,A'\,G^T)_{ik} \\ &= (G\,A'\,G)_{ik}. \end{aligned}$$

# EXAMPLE - CONTRAVARIANT AND MIXED COMPONENTS - PART 3

## Example

We know

$$\mathbf{e}^1 = (4, -4, 1)^T, \qquad \mathbf{e}^2 = (-6, 7, -2)^T, \qquad \mathbf{e}^3 = (-1, 1, 0)^T.$$

Thus

$$\begin{aligned} G = [g^{ik}] &= [\mathbf{e}^i \cdot \mathbf{e}^k] \\ &= \begin{pmatrix} \mathbf{e}^1 \cdot \mathbf{e}^1 & \mathbf{e}^1 \cdot \mathbf{e}^2 & \mathbf{e}^1 \cdot \mathbf{e}^3 \\ \mathbf{e}^2 \cdot \mathbf{e}^1 & \mathbf{e}^2 \cdot \mathbf{e}^2 & \mathbf{e}^2 \cdot \mathbf{e}^3 \\ \mathbf{e}^3 \cdot \mathbf{e}^1 & \mathbf{e}^3 \cdot \mathbf{e}^2 & \mathbf{e}^3 \cdot \mathbf{e}^3 \end{pmatrix} \\ &= \begin{pmatrix} 22 & -54 & -8 \\ -54 & 89 & 13 \\ -8 & 13 & 2 \end{pmatrix} \end{aligned}$$

# EXAMPLE - CONTRAVARIANT AND MIXED COMPONENTS - PART 4

## Example

Thus,

$$\begin{aligned}
[A'^{ik}] &= G\,A'\,G^T \\
&= \begin{pmatrix} 22 & -54 & -8 \\ -54 & 89 & 13 \\ -8 & 13 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 & 11 \\ 1 & 0 & 4 \\ 5 & 2 & 14 \end{pmatrix} \cdot \begin{pmatrix} 22 & -54 & -8 \\ -54 & 89 & 13 \\ -8 & 13 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 38 & 17 & 35 \\ -62 & -28 & -56 \\ -9 & -4 & -8 \end{pmatrix} \cdot \begin{pmatrix} 22 & -54 & -8 \\ -54 & 89 & 13 \\ -8 & 13 & 2 \end{pmatrix} \\
&= \begin{pmatrix} 56 & -84 & -13 \\ -86 & 128 & 20 \\ -17 & 26 & 4 \end{pmatrix}.
\end{aligned}$$

# EXAMPLE - CONTRAVARIANT AND MIXED COMPONENTS - PART 5

## Example

To find the mixed components $A'^{\cdot k}_{i}$ and $A'^{i}_{\cdot k}$, we use the formulas

$$A'^{\cdot k}_{i} = g^{k\ell} A'_{i\ell},$$
$$A'^{i}_{\cdot k} = g^{i\ell} A'_{\ell k}.$$

We see that

$$A'^{\cdot k}_{i} = g^{k\ell} A'_{i\ell} = A'_{i\ell} (G^T)_{\ell k} = (A' G^T)_{ik},$$
$$A'^{i}_{\cdot k} = g^{i\ell} A'_{\ell k} = (G A')_{ik}.$$

# EXAMPLE - CONTRAVARIANT AND MIXED COMPONENTS - PART 6

## Example

Using

$$A'^{\cdot k}_{i} = (A'G^T)_{ik} = (A'G)_{ik},$$

one gets

$$[A'^{\cdot k}_{i}] = A'G = \begin{pmatrix} 4 & 1 & 11 \\ 1 & 0 & 4 \\ 5 & 2 & 14 \end{pmatrix} \cdot \begin{pmatrix} 22 & -54 & -8 \\ -54 & 89 & 13 \\ -8 & 13 & 2 \end{pmatrix}^T$$

$$= \begin{pmatrix} -10 & 16 & 3 \\ 1 & -2 & 0 \\ -55 & 90 & 14 \end{pmatrix}.$$

# EXAMPLE - CONTRAVARIANT AND MIXED COMPONENTS - PART 6

## Example

Using

$$A'^{i}_{\cdot k} = (GA')_{ik},$$

one gets

$$[A'^{i}_{\cdot k}] = (GA')_{ik} = \begin{pmatrix} 22 & -54 & -8 \\ -54 & 89 & 13 \\ -8 & 13 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 & 11 \\ 1 & 0 & 4 \\ 5 & 2 & 14 \end{pmatrix}$$

$$= \begin{pmatrix} 38 & 17 & 35 \\ -62 & -28 & -56 \\ -9 & -4 & -8 \end{pmatrix}.$$

# TENSORS IN GENERALISED COORDINATE SYSTEMS

# THE TRANSFORMATION RULE FOR A VECTOR IN GENERALISED COORDINATE SYSTEM

## The transformation rule

The **transformation rule** for the **covariant** components is

$$A'_i = L^j_{i'} A_j.$$

where

$$L^j_{i'} = \mathbf{e}'_i \cdot \mathbf{e}^j,$$

The **transformation rule** for the **contravariant** components is

$$A'^i = L^{i'}_j A^j$$

where

$$L^{i'}_j = \mathbf{e}_j \cdot \mathbf{e}'^i.$$

# HIGHER-RANK TENSORS - TRANSFORMATIONS

## Higher-rank tensors - transformations

The components of a higher-rank tensor transform similarly to the rank-one case.

However, each index will transform individually.

For instance,

$$A'^{\cdot\,\cdot\,k\ell\,\cdot}_{ij\,\cdot\,\cdot\,m} = \underbrace{L \quad L \quad L \quad L \quad L}_{\text{One transformation matrix for each index}} \quad A$$

# HIGHER-RANK TENSORS - TRANSFORMATIONS

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For instance,

$$A'^{\cdot\cdot k\ell\cdot}_{ij\cdot\cdot m} = \underbrace{L_{i'}\ L_{j'}\ L^{k'}\ L^{\ell'}\ L_{m}}_{\text{Original indices at the same position}} A$$

# HIGHER-RANK TENSORS - TRANSFORMATIONS

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However, each index will transform individually.

For instance,

$$A'^{\cdot\cdot k\ell\cdot}_{ij\cdot\cdot m} = \underbrace{L^{a}_{i'}\ L^{b}_{j'}\ L^{k'}_{c}\ L^{\ell'}_{d}\ L^{e}_{m'}}_{\text{New indices in the remaining positions}} A$$

# HIGHER-RANK TENSORS - TRANSFORMATIONS

## Higher-rank tensors - transformations

The components of a higher-rank tensor transform similarly to the rank-one case.

However, each index will transform individually.

For instance,

$$A'^{\cdot\cdot k\ell\cdot}_{ij\cdot\cdot m} = \qquad L^{a}_{i'}\, L^{b}_{j'}\, L^{k'}_{c}\, L^{\ell'}_{d}\, L^{e}_{m'} \quad \underbrace{A^{\cdot\cdot cd\cdot}_{ab\cdot\cdot e}}_{\text{New indices in the original position}}$$

# SECOND-RANK TENSORS - RELATIONS BETWEEN COMPONENTS

## Second-rank tensors - relations between components

The relations between the various components of a tensor are:

$$
\begin{aligned}
A_{ik} &= g_{i\ell} g_{km} A^{\ell m} = g_{k\ell} A_i^{\cdot \ell} = g_{i\ell} A^{\ell}_{\cdot k},\\
A^{ik} &= g^{i\ell} g^{km} A_{\ell m} = g^{i\ell} A_\ell^{\cdot k} = g^{k\ell} A^{i}_{.\ell},\\
A_i^{\cdot k} &= g^{k\ell} A_{i\ell} = g_{i\ell} A^{\ell k},\\
A^{i}_{.k} &= g^{i\ell} A_{\ell k} = g_{k\ell} A^{i\ell}.
\end{aligned}
$$

# SYMMETRIES

# DEFINITION: SYMMETRIC AND ANTISYMMETRIC TENSORS- CARTESIAN COORDINATES

### Definition.

In **Cartesian coordinates**, a second-rank tensor $T_{ij}$ is **symmetric** if

$$T_{ij} = T_{ji}.$$

### Definition.

In **Cartesian coordinates**, a second-rank tensor $T_{ij}$ is **antisymmetric** if

$$T_{ij} = -T_{ji}.$$

### Example.

We know that the Kronecker delta is symmetric:

$$\delta_{ij} = \delta_{ji}.$$

# DEFINITION: SYMMETRIC AND ANTISYMMETRIC TENSORS IN CARTESIAN COORDINATES

## Definition.

In a **Cartesian** (or orthogonal) coordinate system

- A tensor of rank greater than two **can** be symmetric or antisymmetric with respect to **any pair of indices**.
- We can have for instance that
  - $A_{ijk\ell}$ is symmetric with respect to $j$ and $k$, and
  - $B_{ijk\ell}$ is antisymmetric with respect to $i$ and $\ell$.

  Some tensors might be symmetric/antisymmetric with respect to all indices.

  - For instance, the alternating tensor $\epsilon_{ijk}$ is antisymmetric with respect to any pair of its indices.

# DEFINITION: SYMMETRIC AND ANTISYMMETRIC TENSORS IN GENERALISED COORDINATE SYSTEM

### Definition.

In a **generalised** coordinate system symmetry and antisymmetry apply only to pairs of indices **in the same positions**:

- $A_{ik}^{\cdot\cdot\ell}$ is **symmetric** in $i$ and $k$ if

$$A_{ik}^{\cdot\cdot\ell} = A_{ki}^{\cdot\cdot\ell}.$$

- $B^{ik}_{\cdot\cdot\ell}$ is **anti-symmetric** in $i$ and $k$ if

$$B^{ik}_{\cdot\cdot\ell} = -B^{ki}_{\cdot\cdot\ell}.$$

# MOCK EXAM

## Mock Exam

- In the second lecture next week, I will solve the **Mock Exam**.
- Make sure to solve beforehand, otherwise it **will not be as effective** as preparation for the exam!

# NEXT LECTURE

## Next time...

**Revision**

- Tensor algebra,
- Arc length and the metric tensor,
- Christoffel symbols & Ricci's Theorem,
- Riemann-Christoffel tensor,