Tensor Analysis

Slides week 21 – Lecture 1

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Chapters 2 and 3

Today: Chapter 2–Vector Differential Operators

- Vector differential operators
- Combinations of grad, div and curl
- Grad, div and curl applied to functions

Today: Chapter 3–Local Coordinate Transform

1. Preliminaries,
2. Dual bases,
3. Covariant and contravariant components of a vector,
4. The transformation rule,
5. The relationship between covartiant and contravariant components, and
6. Arc length and the metric tensor.

Reminder

Reminder: Differential operators

Recall that

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right).$$

We have considered three differential operators:

- The gradient ∇f,
- the divergence $\nabla \cdot \mathbf{u}$, and
- the curl $\nabla \times \mathbf{u}$,

where f denotes a scalar field and $\mathbf{u}$ denotes a vector field.

Position vector and differential operators

Position vector

The **position vector** is

$$\mathbf{r} = (x_1, x_2, x_3).$$

We denote its **magnitude** by $\underline{r} = |\mathbf{r}|$.

Position vector and differential operators

We have computed the gradient of $\underline{r}$, and the divergence and the curl of $\mathbf{r}$:

- $\nabla \underline{r} = \frac{\mathbf{r}}{\underline{r}}$,
- $\nabla \cdot \mathbf{r} = 3$, and
- $\nabla \times \mathbf{r} = (0, 0, 0)$.

Summary: Combinations of grad, div, curl

1. Div Grad:

$$\nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x_j \partial x_j} = \nabla^2 f,$$

2. Curl Grad:

$$\nabla \times (\nabla f) = \mathbf{0},$$

3. Grad Div:

$$[\nabla(\nabla \cdot \mathbf{u})]_i = \frac{\partial^2 u_j}{\partial x_i \partial x_j},$$

4. Div Curl:

$$\nabla \cdot (\nabla \times \mathbf{u}) = \mathbf{0},$$

5. Curl Curl:

$$\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}).$$

Grad, div & curl applied to functions

Suffix notation & operating of products of functions

Suffix notation & operating of products of functions

Suffix notation is useful for computing operations on products of functions.

For instance

- scalar products fg,
- vector products $\mathbf{u} \times \mathbf{v}$,
- combinations of the two $f\mathbf{u}$

Next slides

In the following slides we will consider the following operations

1. Gradient of a scalar product $[\nabla(fg)]_i$,
2. Divergence of a scalar vector product $\nabla \cdot (f\mathbf{u})$,
3. Curl of a scalar vector product $\nabla \times (f\mathbf{u})$,
4. Divergence of a vector product $\nabla \cdot (\mathbf{u} \times \mathbf{v})$,
5. Curl of a vector product $\nabla \times (\mathbf{u} \times \mathbf{v})$, and
6. Gradient of a dot product $\nabla(\mathbf{u} \cdot \mathbf{v})$.

1. Gradient of a scalar product

The **gradient** of a scalar product **fg** is

$$
\begin{aligned}
[\nabla(fg)]_i &= \frac{\partial}{\partial x_i}(fg) \\
&= f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i} && \text{(Product rule)} \\
&= [f\nabla g + g\nabla f]_i.
\end{aligned}
$$

Thus,

$$\nabla(fg) = f\nabla g + g\nabla f.$$

This is an extension of the product rule for differentiation.

2. Divergence of a scalar vector product

The **divergence** of a vector product $f\mathbf{u}$ is

$$\begin{aligned}\nabla \cdot (f\mathbf{u}) &= \frac{\partial}{\partial x_i}(f u_i) \\ &= \frac{\partial f}{\partial x_i} u_i + f \frac{\partial u_i}{\partial x_i} \qquad \text{(Product rule)} \\ &= (\nabla f) \cdot \mathbf{u} + f(\nabla \cdot \mathbf{u}).\end{aligned}$$

3. Curl of a scalar vector product

The **curl** of a scalar vector product $f\mathbf{u}$ is

$$
\begin{aligned}
[\nabla \times (f\mathbf{u})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(fu_k) && ((\mathbf{v}\times\mathbf{w})_i = \epsilon_{ijk}v_j w_k) \\
&= \epsilon_{ijk}\frac{\partial f}{\partial x_j}u_k + f\epsilon_{ijk}\frac{\partial u_k}{\partial x_j} && \text{(Product rule)} \\
&= \epsilon_{ijk}\left(\nabla f\right)_j u_k + f\left(\epsilon_{ijk}\frac{\partial u_k}{\partial x_j}\right) \\
&= [\nabla f \times \mathbf{u} + f\,(\nabla \times \mathbf{u})]_i.
\end{aligned}
$$

4. Divergence of a vector product

The **divergence** of a vector product $\mathbf{u}\times\mathbf{v}$ is

$$\begin{aligned}\nabla\cdot(\mathbf{u}\times\mathbf{v}) &= \frac{\partial}{\partial x_i}(\epsilon_{ijk}u_jv_k)\\ &= \epsilon_{ijk}\frac{\partial u_j}{\partial x_i}v_k + \epsilon_{ijk}u_j\frac{\partial v_k}{\partial x_i}\\ &= \left(\epsilon_{kij}\frac{\partial u_j}{\partial x_i}\right)v_k - \left(\epsilon_{jik}\frac{\partial v_k}{\partial x_i}\right)u_j\\ &= (\nabla\times\mathbf{u})\cdot\mathbf{v} - (\nabla\times\mathbf{v})\cdot\mathbf{u}.\end{aligned}$$

5. Curl of a vector product

The **curl** of a vector product $\mathbf{u}\times\mathbf{v}$ is

$$\begin{aligned}
[\nabla \times (\mathbf{u} \times \mathbf{v})]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\mathbf{u} \times \mathbf{v})_k \qquad ((\mathbf{v} \times \mathbf{w})_i = \epsilon_{ijk} v_j w_k) \\
&= \epsilon_{ijk} \frac{\partial}{\partial x_j} (\epsilon_{k\ell m} u_\ell v_m) \\
&= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) \frac{\partial}{\partial x_j} (u_\ell v_m) \\
&= \frac{\partial}{\partial x_j} (u_i v_j) - \frac{\partial}{\partial x_j} (u_j v_i) \qquad (\delta_{ij} a_j = a_i) \\
&= u_i \frac{\partial v_j}{\partial x_j} + v_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial v_i}{\partial x_j} - v_i \frac{\partial u_j}{\partial x_j}.
\end{aligned}$$

5. Curl of a vector product–Part 2

The **curl** of a vector product $\mathbf{u} \times \mathbf{v}$ is

$$[\nabla \times (\mathbf{u} \times \mathbf{v})]_i = u_i \frac{\partial v_j}{\partial x_j} + v_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial v_i}{\partial x_j} - v_i \frac{\partial u_j}{\partial x_j}$$
$$= [\mathbf{u}(\nabla \cdot \mathbf{v})]_i + v_j \frac{\partial u_i}{\partial x_j} - [(\mathbf{u} \cdot \nabla)\mathbf{v}]_i - v_i \frac{\partial u_j}{\partial x_j}.$$

To simplify notation, we define a new operator

$$\mathbf{u} \cdot \nabla = u_j \frac{\partial}{\partial x_j}.$$

We then obtain

$$[\nabla \times (\mathbf{u} \times \mathbf{v})]_i = [\mathbf{u}(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} - \mathbf{v}(\nabla \cdot \mathbf{u})]_i.$$

6. Gradient of a dot product

The **gradient** of a dot product $\mathbf{u} \cdot \mathbf{v}$ was the content of Practical Question 2.4.

More precisely,

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}.$$

Summary Combining differential operators - Part 1

Summary Combining differential operators

1. Gradient of a scalar product

$$\nabla(fg) = f\nabla g + g\nabla f.$$

2. Divergence of a scalar vector product

$$\nabla \cdot (f\mathbf{u}) = (\nabla f) \cdot \mathbf{u} + f(\nabla \cdot \mathbf{u}).$$

3. Curl of a scalar vector product

$$[\nabla \times (f\mathbf{u})]_i = [\nabla f \times \mathbf{u} + f\nabla \times \mathbf{u}]_i.$$

Summary Combining differential operators - Part 2

Summary Combining differential operators - Part 2

4. Divergence of a vector product

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = (\nabla \times \mathbf{u}) \cdot \mathbf{v} - (\nabla \times \mathbf{v}) \cdot \mathbf{u}.$$

5. Curl of a vector product

$$[\nabla \times (\mathbf{u} \times \mathbf{v})]_i = [\mathbf{u}(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} - \mathbf{v}(\nabla \cdot \mathbf{u})]_i.$$

6. Gradient of a dot product

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}.$$

Chapter 3: Preliminaries

Coordinate Systems

Coordinate systems are defined by a set of **basis vectors**.

Vector basis is a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ that

- are **linearly independent:** the only scalars $\lambda_1, \lambda_2, \lambda_3$ satisfying
$$\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \lambda_3 \mathbf{v}_3 = (0, 0, 0)$$
are $\lambda_1 = \lambda_2 = \lambda_3 = 0$.
- **span the space**: Every vector $\mathbf{v}$ can be written as a linear combination
$$\mathbf{v} = \mu_1 \mathbf{v}_1 + \mu_2 \mathbf{v}_2 + \mu_3 \mathbf{v}_3.$$

Example - LI

The following vectors form a basis for $\mathbb{R}^3$:

$$\mathbf{v}_1 = (1,0,0),\ \mathbf{v}_2 = (2,1,0),\ \mathbf{v}_3 = (0,0,1).$$

- **Linearly Independent:** Suppose that

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3 = (0,0,0).$$

Then

$$\begin{aligned}(0,0,0) =& \lambda_1(1,0,0) + \lambda_2(2,1,0) + \lambda_3(0,0,1)\\ &= (\lambda_1,0,0) + (2\lambda_2,\lambda_2,0) + (0,0,\lambda_3)\\ &= (\lambda_1 + 2\lambda_2,\ \lambda_2,\ \lambda_3).\end{aligned}$$

We see that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Thus, these vectors are LI.

Example - Span

Notice that any vector $\mathbf{u} = (u_1, u_2, u_3)$ can be written as

$$\begin{aligned} \mathbf{u} &= (u_1 - 2u_2)(1,0,0) + u_2(2,1,0) + u_3(0,0,1) \\ &= (u_1 - 2u_2)\mathbf{v}_1 + u_2\mathbf{v}_2 + u_3\mathbf{v}_3. \end{aligned}$$

Thus, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span the space.

Orthogonal and Orthonormal coordinate systems

Definitions

- A coordinate system is **orthogonal** if its basis vectors intersect at 90° angles, i.e.,

$$\mathbf{v}_i \cdot \mathbf{v}_j = 0, \ \text{for } i \neq j.$$

- A coordinate system is **orthonormal** if it is orthogonal and its basis vectors have magnitude 1, i.e.,
  - $\mathbf{v}_i \cdot \mathbf{v}_j = 0$, for $i \neq j$, and
  - $|\mathbf{v}_i| = \sqrt{\mathbf{v}_i \cdot \mathbf{v}_i} = 1.$

Cartesian coordinate system

The **Cartesian coordinate** system

$$\mathbf{i} = (1,0,0), \quad \mathbf{j} = (0,1,0), \quad \mathbf{k} = (0,0,1)$$

is an example of **orthonormal coordinate system.**

Example

The coordinate system

$$\mathbf{v} = (1, 1, 0), \quad \mathbf{u} = (-1, 1, 0), \quad \mathbf{w} = (0, 0, 1)$$

is an example of **orthogonal coordinate system:**

- We can show they are LI and span the space.
- We have

$$(1, 1, 0) \cdot (-1, 1, 0) = 1 \cdot (-1) + 1 \cdot 1 + 0 \cdot 0 = 0$$
$$(1, 1, 0) \cdot (0, 0, 1) = 1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0$$
$$(-1, 1, 0) \cdot (0, 0, 1) = -1 \cdot 0 + 1 \cdot 0 + 0 \cdot 1 = 0.$$

However, this is **not** an orthonormal coordinate system because

$$|(1, 1, 0)| = \sqrt{(1, 1, 0) \cdot (1, 1, 0)} = \sqrt{1 \cdot 1 + 1 \cdot 1 + 0 \cdot 0} = \sqrt{2} \neq 1.$$

Definition

Generalised coordinate systems do not necessarily have orthogonal coordinate bases.

Example

In the previous slides we showed the the following are basis vectors

$$\mathbf{v}_1 = (1,0,0),\ \mathbf{v}_2 = (2,1,0),\ \mathbf{v}_3 = (0,0,1).$$

We have

$$(1,0,0)\cdot(2,1,0) = 1\cdot 2 + 0\cdot 1 + 0\cdot 0 = 2 \neq 0.$$

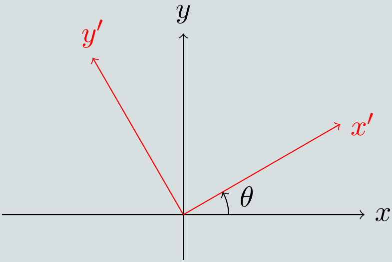
Thus, this is **not** an orthogonal coordinate system (and hence not an orthonormal one).

Cartesian coordinate system in 2D

Define a $2D$ coordinate system by the plane (x_1, x_2).

(For instance, you can think of (x_1, x_2) as two orthogonal axes.)

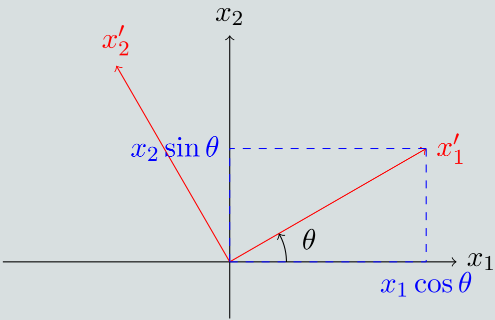
Then, rotate (x_1, x_2) by some angle θ to obtain a new coordinate system (x'_1, x'_2).



Cartesian coordinate system in 2D

Then, any point P in (x_1, x_2) is related to a point in (x'_1, x'_2) via

$$x'_1 = x_1 \cos\theta + x_2 \sin\theta,$$
$$x'_2 = x_2 \cos\theta - x_1 \sin\theta,$$



Cartesian coordinate system in 2D

Again, when we rotate (x_1, x_2) by some angle θ, we obtain a new coordinate system (x_1', x_2'), and any point P in (x_1, x_2) is related to a point in (x_1', x_2') via

$$
\begin{aligned}
x_1' &= x_1 \cos\theta + x_2 \sin\theta, \\
x_2' &= x_2 \cos\theta - x_1 \sin\theta,
\end{aligned}
$$

or in matrix form

$$
\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
$$

Definition.

Define the **rotation matrix** by

$$L = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}.$$

Cartesian coordinate system in 2D

The new coordinate system (x'_1, x'_2) obtained by rotating (x_1, x_2) is

$$x'_1 = L_{11}x_1 + L_{12}x_2 = L_{1j}x_j,$$
$$x'_2 = L_{21}x_1 + L_{22}x_2 = L_{2j}x_j,$$

Or more compactly in suffix notation:

$$x'_i = L_{ij}x_j.$$

Properties of the Rotation Matrix

The rotation matrix

$$L = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

has important properties. For instance, what can we say about the inverse and the transpose of this matrix?

The inverse is simply a rotation through $-\theta$. Thus

$$L^{-1} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = L^T.$$

Properties of the Rotation Matrix

Since $L^T = L^{-1}$, we have $LL^T = I$ and $L^T L = I$, where I is the identity matrix.

In suffix notation we have

$$L_{ij}L^T_{jk} = \delta_{ik} \quad \text{that is} \quad L_{ij}L_{kj} = \delta_{ik}$$
$$L^T_{ij}L_{jk} = \delta_{ik} \quad \text{that is} \quad L_{ji}L_{jk} = \delta_{ik}.$$

Recall the transformation

$$x'_i = L_{ij}x_j.$$

We want to find the inverse transformation. Multiply both sides by L_{ik}:

$$L_{ik}x'_i = L_{ik}L_{ij}x_j = \delta_{kj}x_j = x_k.$$

Properties of the Rotation Matrix

That is, the **inverse transformation** of

$$x'_i = L_{ij}x_j$$

is

$$x_i = L_{ji}x'_j.$$

Remark:

The previous formulas show the following.

- If a vector $\mathbf{v}$ can be writen as (v_1, v_2) in the coordinate system (x_1, x_2), one can use the formula

  $$v'_i = L_{ij}v_j$$

  to find its coordinates (v'_1, v'_2) in (x'_1, x'_2).

- If you know (v'_1, v'_2), you can find (v_1, v_2) by applying the formula

Another Property of the Rotation Matrix

The determinant of the rotation matrix is

$$|L| = \left| \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right| = \cos^2\theta + \sin^2\theta = 1.$$

Cartesian coordinate system in 3D

Question:

What about dimension 3?

Next lecture

- In the next lecture, we will generalise the concept of rotating coordinate systems for the case of dimension 3.
- We will mostly consider this case.
- After that, we will be ready to define a tensor!

Next Lecture

Next time...

- Chapter 3: Local Coordinate Transform.
  - Preliminaries - Part 2